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SCISSORS CONGRUENCES, II*

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1. Introduction

In previous works, Dupont [8], Sah [18, 19], we have indicated interesting connections between Hilbert's Third Problem (suitably extended) and other areas of investigations. The basic open problem is:

Q1. Do the Dehn invariants (appropriately defined and including volume) form a complete system of invariants for the scissors congruence class of polytopes in Euclidean, spherical and hyperbolic n spaces?

This problem is affirmatively settled for $n \leq 4$ in Euclidean spaces (through the work of Sydler [22] and Jessen [10, 11]) and for $n \leq 2$ in the other cases (these are classical). In the present work, we settle some of the questions raised in earlier works.

The first of our result is the following isomorphism:

$$\mathscr{P}(\mathscr{H}^n) \cong \mathscr{P}(\bar{\mathscr{H}}^n), \quad n \geq 2.$$

In general, $\mathscr{P}(X)$ is the scissors congruence group of polytopes in the space X. Unless stated explicitly, the group of motions of X is understood to be the group of all isometries of X. $\bar{\mathscr{H}}^n$ is the extended hyperbolic n-space; it is obtained by adding to the hyperbolic n-space $\mathscr{H}^n$ all the ideal points lying on $\partial\mathscr{H}^n$. The geometry of $\partial\mathscr{H}^n$ is that of conformal geometry on a sphere of dimension $n-1$. The group $\mathscr{P}(\mathscr{H}^n)$ captures the scissors congruence problem in a precise manner. On the other hand, the stable scissors congruence group $\mathscr{P}(\bar{\mathscr{H}}^n)$ is more maneuverable, see Sah [19].

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Along this vein, other types of scissors congruence groups arise. We consider $\mathscr{P}(\partial\mathscr{H}^n)$ defined in a homological manner. This is naturally mapped onto the subgroup $\mathscr{P}(\check{\mathscr{H}}^n)_\infty$ of $\mathscr{P}(\check{\mathscr{H}}^n)$ generated by the totally asymptotic n-simplices. When n is odd, the kernel has exponent dividing 2. For general n, we rederive earlier results concerning $\mathscr{P}(\check{\mathscr{H}}^n)/\mathscr{P}(\check{\mathscr{H}}^n)_\infty$ through homological arguments.

For $n=3$, $\mathscr{P}(\partial\mathscr{H}^3)$ is closely related to scissors congruence type groups considered by S. Bloch, D. Wigner and W. Thurston (independently and all not published): For an arbitrary field F define an abelian group $\mathscr{P}_F$ with generators consisting of all 4-tuples of *distinct* points of the projective line $\mathbb{P}^1(F)$ and with defining relations

$$(gx_0, \dots, gx_3) = (x_0, \dots, x_3), \quad g \in \mathrm{PGL}(2,F),\ x_j \text{ distinct in } \mathbb{P}^1(F);$$

$$\sum_{0\le i\le 4} (-1)^i (x_0, \dots, \hat{x}_i, \dots, x_4) = 0, \quad x_i \text{ distinct in } \mathbb{P}^1(F).$$

Bloch and Wigner essentially obtained an exact sequence of groups involving $H_3(\mathrm{PGL}(2,F),\mathbb{Z})$, $\mathscr{P}_F$, $K_2(F)$ and others (see Theorem 4.10 and Appendix A). Here (and throughout) the homology of groups always means the Eilenberg–MacLane homology groups. In the case of $F=\mathbb{C}$, $\mathscr{P}_\mathbb{C}$ was studied (in a slightly different form) by Thurston in connection with hyperbolic 3-manifolds. Our next principal result is (see Section 5):

$\mathscr{P}_F$ is divisible for an algebraically closed field F.

As a consequence of this and a more careful analysis of Bloch–Wigner's theorem (see Appendix A), we conclude

$H_3(\mathrm{SL}(2,F),\mathbb{Z})$ and $H_3(\mathrm{PSL}(2,F),\mathbb{Z})$ are both divisible when F is an algebraically closed field of characteristic 0.

$\mathscr{P}(\check{\mathscr{H}}^3) = \mathscr{P}(\check{\mathscr{H}}^3)_\infty \cong \mathscr{P}(\mathscr{H}^3)$ is divisible.

At this stage, an obvious open question is:

Q2. Is $\mathscr{P}_F$ uniquely divisible when F is an algebraically closed field?

For fields of characteristic 0, Q2 is equivalent with the following:

Q3. Is $H_3(\mathrm{SL}(2,F),\mathbb{Z})$ the direct sum of $\mathbb{Q}/\mathbb{Z}$ and a $\mathbb{Q}$-vector space when F is an algebraically closed field of characteristic 0?

Affirmative answers to Q2 and Q3 would imply the unique divisibility of $\mathscr{P}(\mathscr{H}^3)$. We note that the absence of torsion in $\mathscr{P}(\mathscr{H}^3)$ would follow from an affirmative answer to Q1 for hyperbolic 3-space. Moreover, the existence of a $\mathbb{Q}$-vector space structure (indeed, an $\mathbb{R}$-vector space structure) was an important step in the proof of the work of Sydler giving rise to an affirmative answer to Q1 for Euclidean 3-space, see Jessen [10] as well as Jessen–Thorup [12] and Sah [18] for details and related problems. However, because of the rigidity result of Cheeger–Simons [6; Proposi-

tion 8.10], see also Cheeger [5] and Dupont [8; Corollary 5.36 and remarks], it is unlikely that $\mathscr{P}_F$ would have an F-vector space structure in analogy with the theorem of Jessen–Thorup.

These questions are related to other conjectures. For example, in connection with the work of Cheeger–Simons, a natural question is:

Q4. Does the invariant $\hat{C}_2$ separate the points of $H_3(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z})$?

Actually, Q4 is only one of an entire family of similar questions. For a discussion of the relation of Q4 with earlier questions, see Dupont [8; Section 6]. In a recent private correspondence, Milnor made the following sweeping conjecture (extending suggestions made by E. Friedlander and others):

Q5. Let G^δ denote a Lie group G with the discrete topology. Let M denote any *finite* G-trivial module. The natural map from G^δ to G then induces isomorphisms $H_*(BG^\delta, M) \cong H_*(BG, M)$.

Here B denotes the classifying space functor so that $H_*(BG^\delta, M)$ is just the Eilenberg–MacLane homology groups of G with coefficient in M.

Conjecture Q5 has been verified by Milnor for solvable groups. The general case can be reduced to the case where G is connected, simple and nonabelian. With these added hypotheses, Q5 is trivial for H_0, H_1. Moreover, Q5 is also valid for many groups on the level of H_2 through K_2 type calculations (more classically, Schur multiplier calculations). Roughly, it is valid for H_2 when G is a quasi-split algebraic group over $\mathbb{R}$ or $\mathbb{C}$ (i.e. G has a Borel subgroup defined over $\mathbb{R}$ or $\mathbb{C}$ respectively). In particular, Q5 is open for H_2 of a compact, simple Lie group of rank >1. For the case of H_2 of $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{C})$ or a split algebraic group over $\mathbb{R}$ or $\mathbb{C}$, see Sah–Wagoner [20]; for the quasi-split case, see Deodhar [7]; for $\mathrm{SO}(3,\mathbb{R})$ and $\mathrm{SU}(2,\mathbb{C})$, we invoke a beautiful theorem of J. Mather asserting that the inclusion of the circle group into $\mathrm{SU}(2,\mathbb{C})$ induces a surjective map on H_2. Our result implies Q5 for H_3 of $\mathrm{SL}(2,\mathbb{C})$.

We note that Q3 is a special case of Q4 as well as Q5. Moreover, the validity of Q5 would imply the characteristic 0 version of a conjecture attributed to Lichtenbaum by Quillen in his Vancouver International Congress talk:

Q6. Let $p>0$ be a prime distinct from the characteristic of the algebraically closed field F. The cohomology ring $H^*(\mathrm{BGL}(F), \mathbb{F}_p)$ is a polynomial ring over $\mathbb{F}_p$ with generators c_i of degree $2i$, $i \geq 1$.

Our results are consistent with all these conjectures.

As another example consistent with these conjectures, we show that a part of the scissors congruence group in the spherical case arising from fundamental domains of finite groups acting isometrically on spheres is in fact isomorphic to $\mathbb{Q}$ so that it is

uniquely divisible. This part is responsible for a known $\mathbb{Q}/\mathbb{Z}$ direct summand in $H_3(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z})$.

2. Scissors congruence in hyperbolic and extended hyperbolic space

As in Sah [19; Section 3] we let $\mathscr{P}(\mathscr{H}^n)$ and $\mathscr{P}(\check{\mathscr{H}}^n)$ denote the scissors congruence groups for hyperbolic n-space and the extended hyperbolic n-space respectively. The main result of this section is the following:

Theorem 2.1. *The natural inclusion $\mathscr{H}^n \subset \check{\mathscr{H}}^n$ induces an isomorphism*

$$\iota_n : \mathscr{P}(\mathscr{H}^n) \xrightarrow{\cong} \mathscr{P}(\check{\mathscr{H}}^n), \quad n > 1.$$

For the proof of this theorem we shall use the homological approach described in Dupont [8; Section 6]. Thus let $\mathscr{T}(\mathscr{H}^n)$ be the Tits complex of flags of proper geodesic subspaces of $\mathscr{H}^n$. This has the homology concentrated in dimension $n-1$ and we put

$$\mathrm{St}(\mathscr{H}^n) = \tilde{H}_{n-1}(\mathscr{T}(\mathscr{H}^n), \mathbb{Z})$$

considered as a module for the group $G(n)$ of all isometries of $\mathscr{H}^n$ (St = 'Steinberg module'). Using the usual orientation of $\mathscr{H}^n$ there is a natural isomorphism

$$\mathscr{P}(\mathscr{H}^n) \cong H_0(G(n), \mathrm{St}(\mathscr{H}^n)^t) \tag{2.2}$$

where the upper index ^{t} signifies that the action of $G(n)$ on $\mathrm{St}(\mathscr{H}^n)$ is twisted by the determinant ($=\pm 1$). Similarly

$$\mathscr{P}(\check{\mathscr{H}}^n) \cong H_0(G(n), \mathrm{St}(\check{\mathscr{H}}^n)^t) \tag{2.3}$$

with

$$\mathrm{St}(\check{\mathscr{H}}^n) \cong \tilde{H}_{n-1}(\mathscr{T}(\check{\mathscr{H}}^n), \mathbb{Z})$$

where $\mathscr{T}(\check{\mathscr{H}}^n)$ is the larger complex of flags in which points are allowed to lie on the boundary $\partial\mathscr{H}^n$ of $\mathscr{H}^n$. Under the isomorphisms (2.2) and (2.3) the map ι_n clearly corresponds to the map induced by the natural inclusion

$$i : \mathscr{T}(\mathscr{H}^n) \subset \mathscr{T}(\check{\mathscr{H}}^n).$$

Also for $p \in \partial\mathscr{H}^n$ let $\mathscr{T}(\mathscr{H}^n, p)$ be the complex of flags of proper subspaces of $\mathscr{H}^n$ going through the ideal point p. Taking the upper half space model for $\mathscr{H}^n$ and $p = \infty$ we identify $\partial\mathscr{H}^n$ with the boundary $\mathbb{R}^{n-1} \cup \{\infty\}$. Then by cutting a geodesic subspace through ∞ with $\partial\mathscr{H}^n$ we obtain an obvious isomorphism of $\mathscr{T}(\mathscr{H}^n, p)$ with the *affine Tits complex* $\mathscr{AT}(\mathbb{R}^{n-1})$ of flags of proper affine subspaces of $\mathbb{R}^{n-1}$. Also this complex has the homology concentrated in a single dimension $n-2$, cf. Dupont [8; proof of Proposition 5.3], so we define for $n \geq 2$

$$\mathrm{St}(\mathscr{H}^n, p) = \tilde{H}_{n-2}(\mathscr{T}(\mathscr{H}^n, p), \mathbb{Z}), \quad n \geq 2,$$

$$\mathrm{ASt}(\mathbb{R}^{n-1}) = \tilde{H}_{n-2}(\mathscr{AT}(\mathbb{R}^{n-1}), \mathbb{Z}), \quad n \geq 2,$$

where the first group is considered as a module for the isotropy group $G(n)_p$ at $p \in \partial\mathscr{H}^n$ and the second is a module for the group of affine transformations of $\mathbb{R}^{n-1}$. In the upper halfspace model $G(n)_\infty$ acts on $\mathbb{R}^{n-1} \subset \partial\mathscr{H}^n$ as the group $\mathrm{Sim}(n-1)$ of *similarities* of $\mathbb{R}^{n-1}$, i.e. affine transformations which multiply distances by some positive scalar. Thus we have:

Lemma 2.4. *There is an exact sequence of $G(n)$-modules for $n \geq 2$*

$$0 \longrightarrow \mathrm{St}(\mathscr{H}^n) \xrightarrow{i} \mathrm{St}(\bar{\mathscr{H}}^n) \longrightarrow \coprod_{p \in \partial\mathscr{H}^n} \mathrm{St}(\mathscr{H}^n, p) \longrightarrow 0.$$

Here for $p = \infty$ the isotropy group $G(n)_\infty \cong \mathrm{Sim}(n-1)$ and $\mathrm{St}(\mathscr{H}^n, p) \cong \mathrm{ASt}(\mathbb{R}^{n-1})$.

Proof. In order to establish the exact sequence we use the long exact homology sequence for the pair $(\mathscr{T}(\bar{\mathscr{H}}^n), \mathscr{T}(\mathscr{H}^n))$ and the obvious isomorphism of chain complexes

$$C_*(\mathscr{T}(\bar{\mathscr{H}}^n), \mathscr{T}(\mathscr{H}^n)) \cong \coprod_{p \in \partial\mathscr{H}^n} C_*(\mathscr{T}(\mathscr{H}^n), p).$$

Then the result just follows from the fact mentioned above that both $\mathscr{T}(\mathscr{H}^n)$ and $\mathscr{T}(\mathscr{H}^n, p)$ have the homology concentrated in a single dimension. The second statement of the lemma is obvious. □

Next we study the group homology $H_*(\mathrm{Sim}(n), \mathrm{ASt}(\mathbb{R}^n)^t)$ where as usual the action on $\mathrm{ASt}(\mathbb{R}^n)$ is twisted by the determinant. Notice that $\mathrm{Sim}(n)$ is a semidirect product

$$\mathrm{Sim}(n) = T(n) \cdot \mathrm{Sim}_0(n)$$

where $T(n)$ is the group of translations (i.e. $T(n)$ is the additive group of $\mathbb{R}^n$) and where $\mathrm{Sim}_0(n)$ consists of linear similarities (i.e. fixing 0). Here

$$\mathrm{Sim}_0(n) \cong O(n, \mathbb{R}) \times \mathbb{R}_+^\times$$

where $\lambda \in \mathbb{R}_+^\times$ corresponds to the dilatation μ_λ given by $\mu_\lambda(x) = \lambda x$, $x \in \mathbb{R}^n$. We now have:

Lemma 2.5. *The inclusion $\mathrm{Sim}_0(n) \subset \mathrm{Sim}(n)$ induces an isomorphism*

$$H_*(\mathrm{Sim}_0(n), \mathbb{Z}^t) \xrightarrow{\cong} H_*(\mathrm{Sim}(n), \mathbb{Z}^t).$$

Proof. The Hochschild–Serre spectral sequence for the split extension

$$0 \longrightarrow T(n) \longrightarrow \mathrm{Sim}(n) \longrightarrow \mathrm{Sim}_0(n) \longrightarrow 1$$

has the E^2-term

$$E^2_{*,q} = H_*(\mathrm{Sim}_0(n), H_q(T(n), \mathbb{Z})^\iota), \quad q = 0, 1, 2, \ldots.$$

Since $T(n) \cong \mathbb{R}^n$ we have (cf. Dupont [8; Lemma 3.1])

$$H_q(T(n), \mathbb{Z}) \cong \Lambda^q_{\mathbb{Z}}(\mathbb{R}^n)$$

and for $\lambda \in \mathbb{Q}^\times_+$ the induced action by μ_λ is given by multiplication by λ^q. Now μ_λ lies in the center of $\mathrm{Sim}_0(n)$ so it follows from the 'center kills' lemma (cf. Sah [17; Proposition 2.7c]) that

$$E^2_{*,q} = H_*(\mathrm{Sim}_0(n), \Lambda^q_{\mathbb{Z}}(\mathbb{R}^n)^\iota) = 0, \quad q = 1, 2, \ldots.$$

Therefore

$$H_*(\mathrm{Sim}(n), \mathbb{Z}^\iota) \cong E^2_{*,0} = H_*(\mathrm{Sim}_0(n), \mathbb{Z}^\iota)$$

where the isomorphism is given by the 'edge'-homomorphism. The inverse is induced by inclusion because we started with a split extension of groups. □

Corollary 2.6. *For $n > 0$,*

$$H_*(\mathrm{Sim}(n), \mathrm{ASt}(\mathbb{R}^n)^\iota) = 0.$$

Proof. We proceed by induction: For $n = 1$ consider the exact sequence of $\mathrm{Sim}(1)$-modules

$$0 \longrightarrow \mathrm{ASt}(\mathbb{R}^1) \longrightarrow \coprod_{p \in \mathbb{R}^1} \mathbb{Z}(p) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \tag{2.7}$$

where ε is the augmentation to the trivial module. By Shapiro's lemma (see Cartan–Eilenberg [4; Chapter X, Proposition 7.4])

$$H_*(\mathrm{Sim}(1), \{\coprod_{p \in \mathbb{R}^1} \mathbb{Z}(p)\}^\iota) \cong H_*(\mathrm{Sim}_0(1), \mathbb{Z}^\iota)$$

and it is easily seen that the induced map by ε corresponds to the map induced by the inclusion of groups. Hence $H_*(\mathrm{Sim}(1), \mathrm{ASt}(\mathbb{R}^1)^\iota) = 0$ follows from the exact homology sequence for the coefficient sequence (2.7).

For $n > 1$ consider the Lusztig exact sequence of modules for $\mathrm{Sim}(n)$ (cf. Dupont [8; proof of Proposition 5.24])

$$0 \longrightarrow \mathrm{ASt}(\mathbb{R}^n) \longrightarrow \coprod_{V^{n-1}} \mathrm{ASt}(V^{n-1}) \longrightarrow \cdots \longrightarrow \coprod_{V^0} \mathrm{ASt}(V^0) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \tag{2.8}$$

where V^j runs through all j-dimensional *affine subspaces of* $\mathbb{R}^n$. Since the stabilizer $\mathrm{Sim}(n)_{\mathbb{R}^j}$ of $\mathbb{R}^j \subset \mathbb{R}^n$ is a product

$$\mathrm{Sim}(n)_{\mathbb{R}^j} \cong \mathrm{Sim}(j) \times O(n-j, \mathbb{R}),$$

it follows using Shapiro's lemma, the inductive hypothesis and the Künneth theorem that

$$H_*(\mathrm{Sim}(n), \{\coprod_{V^j} \mathrm{ASt}\, V^j\}^\iota) \cong H_*(\mathrm{Sim}(j) \times O(n-j, \mathbb{R}), \mathrm{ASt}(\mathbb{R}^j)^\iota \otimes \mathbb{Z}^\iota) = 0$$

for $0<j<n$. Similarly

$$H_*(\mathrm{Sim}(n), \{\coprod_{V^0} \mathrm{ASt}(V^0)\}^\iota) \cong H_*(\mathrm{Sim}_0(n), \mathbb{Z}^\iota)$$

and the map induced by $\varepsilon : \coprod_{V^0} \mathrm{ASt}(V^0) \to \mathbb{Z}$ corresponds again to the map induced by the inclusion $\mathrm{Sim}_0(n) \subset \mathrm{Sim}(n)$. Therefore if we split the Lusztig exact sequence (2.8) into short exact sequences

$$0 \longrightarrow Z_0 \longrightarrow \coprod_{V^0} \mathrm{ASt}(V^0) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

$$\cdots$$

$$0 \longrightarrow Z_j \longrightarrow \coprod_{V^j} \mathrm{ASt}(V^j) \longrightarrow Z_{j-1} \longrightarrow 0,$$

$$\cdots$$

$$0 \longrightarrow \mathrm{ASt}(\mathbb{R}^n) \longrightarrow \coprod_{V^{n-1}} \mathrm{ASt}(V^{n-1}) \longrightarrow Z_{n-2} \longrightarrow 0,$$

then we conclude that

$$H_k(\mathrm{Sim}(n), \mathrm{ASt}(\mathbb{R}^n)^\iota) \cong H_{k+1}(\mathrm{Sim}(n), Z_{n-2}^\iota)$$

$$\cong \cdots \cong H_{k+n-1}(\mathrm{Sim}(n), Z_0^\iota) = 0$$

for all $k = 0, 1, 2, \ldots$. □

Proof of Theorem 2.1. By (2.2), (2.3) and Lemma 2.4 it suffices to show

$$H_*(G(n), \{\coprod_{p \in \partial \mathscr{H}^n} \mathrm{St}(\mathscr{H}^n, p)^\iota\}) = 0 \tag{2.9}$$

(in fact, $* = 0$, 1 would be enough). Again by Shapiro's lemma the left-hand side of (2.9) is isomorphic to

$$H_*(G(n)_\infty, \mathrm{St}(\mathscr{H}^n, \infty)^\iota) \cong H_*(\mathrm{Sim}(n-1), \mathrm{ASt}(\mathbb{R}^{n-1})^\iota) = 0$$

for $n > 1$ by Corollary 2.6. □

Remark. Notice that the argument used in Sah [19; Proposition 3.3] for the surjectivity of ι_n is essentially the same as the above except for the homological formulation.

3. Hyperbolic scissors congruence with only infinite vertices

In relation to $\mathscr{P}(\mathscr{H}^n)$ it is natural to consider a scissors congruence group $\mathscr{P}(\partial \mathscr{H}^n)$ generated by totally asymptotic polytopes, i.e. polytopes with all vertices lying on the boundary $\partial \mathscr{H}^n$. As mentioned in Sah [19; Appendix 1], there are at least two reasonable choices for a definition, and at the moment we do not know if they are equivalent. For our purpose the one indicated in Dupont [8; Section 2, Remark 2] is convenient since it allows an analysis using homological algebra. Thus we define $\mathscr{P}(\partial \mathscr{H}^n)$ to be the abelian group generated by all $(n+1)$-tuples $(a_0, \ldots, a_n)$ of points $a_i \in \partial \mathscr{H}^n$ subject to the relations

(3.1)(i) $(a_0, \dots, a_n) = 0$ if all a_i's lie in a geodesic subspace of dimension less than n.

(3.1)(ii) $\sum_{0 \le i \le n+1} (-1)^i (a_0, \dots, \hat{a}_i, \dots, a_{n+1}) = 0$, $a_i \in \partial \mathscr{H}^n$ arbitrary.

(3.1)(iii) $(ga_0, \dots, ga_n) = \det(g) \cdot (a_0, \dots, a_n)$, $a_i \in \partial \mathscr{H}^n$ arbitrary and $g \in G(n) =$ group of all isometries of $\mathscr{H}^n$.

Notice that if $a_0, \dots, a_n$ lie in a geodesic hyperplane then $2(a_0, \dots, a_n) = 0$ already follows from (iii). Thus, apart from some possible 2-torsion, (i) is unnecessary.

In this context, Thurston (unpublished) has studied a similar group $\mathscr{P}'(\partial \mathscr{H}^n)$ (for $n = 3$) where (i) and (iii) of (3.1) are respectively replaced by

(i') $(a_0, \dots, a_n) = 0$ if $a_i = a_j$ for some $i \neq j$,

(iii') $(a_0, \dots, a_n) = (ga_0, \dots, ga_n)$, $a_i \in \partial \mathscr{H}^n$ arbitrary and g any *orientation preserving* isometry of $\mathscr{H}^n$.

Actually as we shall see later (see Remark after Corollary 4.7) $\mathscr{P}'(\partial \mathscr{H}^3)$ is the group $\mathscr{P}_{\mathbb{C}}$ defined in the introduction. Notice that a priori $\mathscr{P}'(\partial \mathscr{H}^3)$ is only a quotient of $\mathscr{P}_{\mathbb{C}}$ since the a_i's in (3.1)(ii) need not be distinct.

For the relationship between $\mathscr{P}(\partial \mathscr{H}^n)$ and $\mathscr{P}'(\partial \mathscr{H}^n)$ we need the following notation.

In general, let A be a module for the cyclic group $\langle \tau \rangle$ of order 2. For $\varepsilon = \pm$, let $A^{\varepsilon\tau}$ or A^{ε} denote $H^0(\langle \varepsilon\tau \rangle, A)$ and call it the ε-eigenspace of A for τ. Similarly, let $A_{\varepsilon\tau}$ or A_{ε} denote $H_0(\langle \varepsilon\tau \rangle, A)$ and call it the ε-coeigenspace of A for τ. Thus $A_{\varepsilon} = A/(1 - \varepsilon\tau)A$ is the largest quotient group of A on which τ acts according to $\varepsilon \cdot \mathrm{Id}$ while A^{ε} is the largest subgroup of A on which τ acts according to $\varepsilon \cdot \mathrm{Id}$. If we let $\mathscr{C}_2$ denote the Serre class of 2-primary abelian groups of finite exponent (cf. Dupont [8; Definition 5.25] then $A^{\varepsilon} \cong A_{\varepsilon} \bmod \mathscr{C}_2$ is given by the natural map from A^{ε} to A_{ε}. This map is injective when A has no 2-torison and is surjective when A is 2-divisible. Similarly, $A \cong A^+ \amalg A^- \cong A_+ \amalg A_- \bmod \mathscr{C}_2$.

Notice that the subgroup of all orientation preserving isometries of $\mathscr{H}^n$ has index 2 in $G(n)$ and a coset representative can be taken to be any reflection τ with respect to a geodesic hyperplane. We then have

$$\mathscr{P}(\partial \mathscr{H}^n) \cong \mathscr{P}'(\partial \mathscr{H}^n)_- \bmod \mathscr{C}_2. \tag{3.2}$$

As we shall see in (5.24), mod $\mathscr{C}_2$ is not necessary when $n = 3$.

To familiarize ourselves, we consider the cases $n = 1$ or 2.

$G(1)$ is a cyclic group of order 2 and exchanges the two points of $\partial \mathscr{H}^1$. $\mathscr{P}'(\partial \mathscr{H}^1) \cong \mathbb{Z}G(1)$, the group ring of $G(1)$. $\mathscr{P}(\partial \mathscr{H}^1) \cong \mathscr{P}'(\partial \mathscr{H}^1)^- \cong \mathscr{P}'(\partial \mathscr{H}_1)_- \cong \mathbb{Z}$.

$G(2) \cong \mathrm{PGL}(2, \mathbb{R})$ and $\partial \mathscr{H}^2$ is identified with $\mathbb{P}^1(\mathbb{R})$ in an equivariant manner. In this case, (i) and (i') are equivalent. Using 3-transitivity of $\mathrm{PGL}(2, \mathbb{R})$ on $\mathbb{P}^1(\mathbb{R})$ together with the orientation of $\mathbb{P}^1(\mathbb{R})$, it is easy to see that $\mathscr{P}'(\partial \mathscr{H}^2) \cong \mathbb{Z}[\mathrm{PGL}(2, \mathbb{R})/\mathrm{PSL}(2, \mathbb{R})]$, the group ring again. Similarly, $\mathscr{P}(\partial \mathscr{H}^2) \cong \mathscr{P}'(\partial \mathscr{H}^2)^- \cong \mathscr{P}'(\partial \mathscr{H}^2)_- \cong \mathbb{Z}$.

We next study the natural map induced by the inclusion $\partial \mathscr{H} \subset \mathscr{H}^n$:

$$\kappa_n : \mathscr{P}(\partial \mathscr{H}^n) \to \mathscr{P}(\mathscr{H}^n).$$

As in Section 2, we have a Tits complex $\mathscr{T}(\partial \mathscr{H}^n)$ of flags of proper geodesic

subspaces where the only 0-dimensional ones are points of $\partial\mathscr{H}^n$. Again

$$\mathrm{St}(\partial\mathscr{H}^n) = H_{n-1}(\mathscr{T}(\partial\mathscr{H}^n), \mathbb{Z}) \tag{3.3}$$

is the only non-zero homology group and there is a natural isomorphism

$$\mathscr{P}(\partial\mathscr{H}^n) \cong H_0(G(n), \mathrm{St}(\partial\mathscr{H}^n)^t) \tag{3.4}$$

such that the map κ_n corresponds via (3.4) and (2.3) to the induced map in homology for the inclusion

$$k: \mathscr{T}(\partial\mathscr{H}^n) \subset \mathscr{T}(\bar{\mathscr{H}}^n).$$

Again similar to Lemma 2.4 we have:

Lemma 3.5. *There is an exact sequence of $G(n)$-modules for $n \geq 2$:*

$$0 \longrightarrow \mathrm{St}(\partial\mathscr{H}^n) \xrightarrow{k} \mathrm{St}(\bar{\mathscr{H}}^n) \longrightarrow \coprod_{p \in \mathscr{H}^n} \mathrm{St}(\mathscr{H}^n, p) \longrightarrow 0.$$

In this case, the isotropy group $G(n)_p$ is just the orthogonal group in the tangent space $T_p(\mathscr{H}^n)$ and

$$\mathrm{St}(\mathscr{H}^n, p) \cong \mathrm{St}(T_p(\mathscr{H}^n))$$

is the classical Steinberg module for this vector space (cf. Dupont [8; Definition 5.2]). Hence by Shapiro's lemma

$$H_*(G(n), \{\coprod_{p \in \mathscr{H}^n} \mathrm{St}(\mathscr{H}^n, p)\}^t) \cong H_*(O(n, \mathbb{R}), \mathrm{St}(\mathbb{R}^n)^t) \tag{3.6}$$

and in particular for *n odd* this group has only elements of order 2 by the 'center kills'-lemma (cf. Dupont [8; Remark 2 following Corollary 5.18). We therefore obtain from Lemma 3.5:

Proposition 3.7. (i) *There is an exact sequence*

$$\longrightarrow H_1(O(n, \mathbb{R}), \mathrm{St}(\mathbb{R}^n)^t) \longrightarrow \mathscr{P}(\partial\mathscr{H}^n) \xrightarrow{\kappa} \mathscr{P}(\bar{\mathscr{H}}^n) \longrightarrow H_0(O(n, \mathbb{R}), \mathrm{St}(\mathbb{R}^n)^t) \longrightarrow$$

(ii) *In particular, for n* ***odd****, the map*

$$\kappa_n: \mathscr{P}(\partial\mathscr{H}^n) \longrightarrow \mathscr{P}(\bar{\mathscr{H}}^n)$$

is surjective with kernel consisting of elements of order at most 2.

Remarks. 1. In (ii) κ_n, *n odd*, is surjective (not just surjective mod $\mathscr{C}_2$) since $\mathscr{P}(\bar{\mathscr{H}}^n)$ is 2-divisible by a classical argument (cf. Sah [18; Proposition 1.4.3, p. 17]).

2. By Dupont [8; Corollary 5.18]

$$H_0(O(n, \mathbb{R}), \mathrm{St}(\mathbb{R}^n)^t) \cong \mathscr{P}(S(\mathbb{R}^n))/2[\text{point}] * \mathscr{P}(S(\mathbb{R}^{n-1}))$$

(in the notation of Sah[18]); therefore Proposition 3.7 (i) reproves Sah [19; Proposition 3.7].

3. For n even, $H_1(O(n,\mathbb{R}), \mathrm{St}(\mathbb{R}^n)^t)$ can be studied via the Lusztig exact sequence as in Dupont [8; Section 5, in particular (5.23)]. However already for $n=4$ this involves $H_4(SU(2,\mathbb{C}),\mathbb{Z})$ about which little is known.

4. In view of Q1 in Section 1, it seems reasonable to conjecture the injectivity of κ_n.

4. Dehn invariants and exact sequences

In this section we shall restrict to the case $\mathscr{H}^3$ where the classical hyperbolic Dehn invariant exists and in particular we shall study the extension of this to $\check{\mathscr{H}}^3$ and its relation to an interesting exact sequence due to S. Bloch and D. Wigner (unpublished notes).

Since by Theorem 2.1 $\mathscr{P}(\mathscr{H}^3) \cong \mathscr{P}(\check{\mathscr{H}}^3)$, the classical Dehn invariant Ψ ($=H\Psi^{(2)}$ in the notation of Sah [18])

$$\Psi : \mathscr{P}(\mathscr{H}^3) \longrightarrow \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$$

extends uniquely to a map $\Psi : \mathscr{P}(\check{\mathscr{H}}^3) \to \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$. As mentioned in Sah [19; Appendix 2], a direct geometric definition was given by Thurston as follows: Recall that for $P \subset \mathscr{H}^3$ a polyhedron with only *finite* vertices the classical definition is

$$\Psi(P) = \sum_A l(A) \otimes \theta_A \tag{4.1}$$

where A runs through all edges of P, $l(A)$ is the length and θ_A is the dihedral angle at A (divided by 2π). Now if P has vertex v at infinity we delete a horoball around v, and for A an edge ending at v the length $l(A)$ is measured only up to the horosphere. The indeterminacy in this definition vanishes since the sum of the angles at edges ending at v is a multiple of π. In particular $\Psi(P)$ is defined for totally asymptotic polyhedra and by Proposition 3.7 Ψ is actually determined by the values on such polyhedra. Let us compute Ψ explicitly on some special simplices in $\check{\mathscr{H}}^3$:

As in Sah [19; Section 4] let $A(\alpha,\beta,\gamma)$ denote any totally asymptotic 3-simplex with dihedral angles (around one vertex in the positive orientation) $\alpha,\beta,\gamma > 0$ with $\alpha+\beta+\gamma=\pi$. We recall that in $\mathscr{P}(\check{\mathscr{H}}^3)$

$$[A(\alpha,\beta,\gamma)] = \mathscr{L}(\alpha) + \mathscr{L}(\beta) + \mathscr{L}(\gamma) \tag{4.2}$$

where $\mathscr{L}(\theta)$, $0<\theta<\pi/2$, is a simplex with 3 infinite vertices such that

$$2\mathscr{L}(\theta) = \left[A\left(2\theta, \frac{\pi}{2}-\theta, \frac{\pi}{2}-\theta\right)\right], \quad 0<\theta<\pi/2. \tag{4.3}$$

Lemma 4.4. (i) $\Psi(\mathscr{L}(\theta)) = 2\log 2\sin\theta \otimes (\theta/2\pi)$, $0<\theta<\pi/2$.

(ii) $\Psi(A(\alpha,\beta,\gamma)) = \log 2\sin\alpha \otimes (\alpha/\pi) + \log 2\sin\beta \otimes (\beta/\pi) + \log 2\sin\gamma \otimes (\gamma/\pi)$.

Proof. (ii) follows from (i) and (4.2). Using (4.3), we consider $A(2\theta,\theta',\theta')$, $\theta+\theta'=\pi/2$, in the upper half space model. With one of the vertices at ∞, the other

three can be taken to lie on the unit circle in the horizontal plane. These three base vertices then form an isosocles Euclidean triangle with apex angle 2θ. Using the degree 2 symmetry of the base, we may construct horospheres corresponding to Euclidean spheres tangent to the horizontal plane with centers at Euclidean distances of $(\cot\theta)/2$, $\sin 2\theta$ and $\sin 2\theta$ above the three base vertices. Simple Euclidean geometry arguments show that these three horospheres are pairwise tangent to each other with points of tangency lying on the base edges (corresponding to semicircles orthogonal to the horizontal plane at the base vertices). The horosphere about ∞ can be chosen to be any high horizontal plane. When θ is small, it can be chosen to be at the Euclidean distance $\cot\theta$ above; when θ is close to $\pi/2$, it can be chosen to be at the Euclidean distance $2\sin 2\theta$ above. In the first case, we have two equal lengths each over the base vertex with angle θ'. In the second case, we have one length over the base vertex with angle 2θ. In both cases, we need to calculate the hyperbolic distance between points on a vertical line at Euclidean distances $\cot\theta$ and $2\sin 2\theta$ above the horizontal plane. In the tensor product, $\pi/2$ can be replaced by 0. As a result, the two possible cases gives the same answer and we may as well concentrate on the second case. The hyperbolic distance is then

$$\log 2\sin 2\theta - \log\cot\theta = \log(2\sin\theta)^2 = 2\log 2\sin\theta.$$

It follows that

$$\Psi(2\mathscr{L}(\theta)) = 2\log 2\sin\theta \otimes (2\theta/2\pi).$$

Since Ψ is additive and has value in a $\mathbb{Q}$-vector space, division by 2 leads to the desired result. We leave the diagrams in Fig. 1 for the readers to check the argument. □

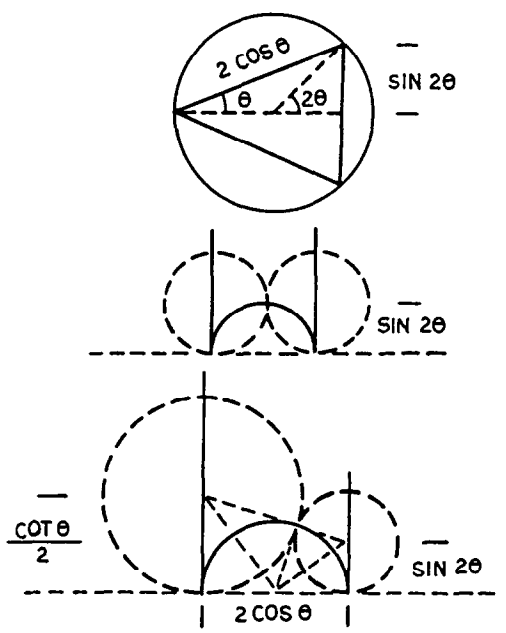


Fig. 1.

Next we immediately obtain from Dupont [8; Theorem 1.4] together with our Theorem 2.1 and Proposition 3.7:

Proposition 4.5. *There is an exact sequence* mod $\mathscr{C}_2$

$$0 \longrightarrow H_3(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z})_- \longrightarrow \mathscr{P} \xrightarrow{\Psi} \mathbb{R}\otimes\mathbb{R}/\mathbb{Z} \longrightarrow H_2(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z})_- \longrightarrow 0$$

where $\mathscr{P} = \mathscr{P}(\mathscr{H}^3)$, $\mathscr{P}(\bar{\mathscr{H}}^3)$ *or* $\mathscr{P}(\partial\mathscr{H}^3)$, Ψ *is the Dehn invariant and* $_-$ *indicates the* (-1)*-coeigenspace for complex conjugation.*

Remark. Notice that $\mathscr{P} = \mathscr{P}(\bar{\mathscr{H}}^3)$ is generated by the set of elements $\mathscr{L}(\theta)$, $0<\theta<\pi/2$, (because of Proposition 3.7 and (4.2)). Therefore Proposition 4.5 implies the result of Sah–Wagoner [20; Proposition 1.23] that $K_2(\mathbb{C})^-$ $(=H_2(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z})^-)$ is the quotient of $\mathbb{R}\otimes\mathbb{R}/\mathbb{Z}$ by the subgroup generated by all elements of the form

$$\log 2\sin\theta\otimes(\theta/2\pi), \quad 0<\theta<\pi/2.$$

The exact sequence in Proposition 4.5 has an analogue involving $\mathscr{P}'(\partial\mathscr{H}^3)$ or $\mathscr{P}_{\mathbb{C}}$ which we shall describe next. The resulting sequence is originally due to Bloch and Wigner (unpublished notes, see also Bloch [2]). As a model for $\mathscr{H}^3$ we take the upper half space bounded by the Riemann sphere $\partial\mathscr{H}^3 = \mathbb{P}^1(\mathbb{C}) = \mathbb{C}\cup\{\infty\}$. Notice that the group of all orientation preserving isometries of $\mathscr{H}^3$ is isomorphic to $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{PGL}(2,\mathbb{C})$ which acts on $\partial\mathscr{H}^3$ through the usual fractional linear action of $\mathrm{PGL}(2,\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$:

$$g(z) = (az+b)/(cz+d), \quad z\in\mathbb{C}\cup\{\infty\}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This is one of the few exceptional isomorphisms among classical groups. Recall the definition of the *cross ratio:*

$$\{a_0 : a_1 : a_2 : a_3\} = (a_0-a_2)(a_1-a_3)/(a_0-a_3)(a_1-a_2) \in \mathbb{C}-\{0,1\}$$

for four distinct points $a_0, a_1, a_2, a_3 \in \mathbb{P}^1(\mathbb{C})$. Our definition is chosen so that the cross ratio of $\infty, 0, 1, z$ is just z. We recall the well known fact (valid for any field F in place of $\mathbb{C}$ provided that we use $\mathrm{PGL}(2,F)$ in place of $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{PGL}(2,\mathbb{C})$):

Proposition 4.6. (i) $\mathrm{PSL}(2,\mathbb{C})$ *acts exactly* 3*-transitively on* $\mathbb{P}^1(\mathbb{C})$.

(ii) *For two quadruples* (a_0,a_1,a_2,a_3) *and* (a_0',a_1',a_2',a_3') *of distinct points there exists* $g\in\mathrm{PSL}(2,\mathbb{C})$ *with* $(ga_0, ga_1, ga_2, ga_3) = (a_0', a_1', a_2', a_3')$ (g *is necessarily unique*) *if and only if*

$$\{a_0' : a_1' : a_2' : a_3'\} = \{a_0 : a_1 : a_2 : a_3\}.$$

An obvious consequence is:

Corollary 4.7. (i) $\mathscr{P}'(\partial\mathscr{H}^3)$ *is the abelian group generated by* $\{z\} = [(\infty, 0, 1, z)]$,

$z \in \mathbb{C} - \{0, 1\}$ *subject to the relations*

$$\sum_{0 \le j \le 4} (-1)^j \{\{a_0 : \cdots : \hat{a}_j : \cdots : a_4\}\} = 0, \quad a_j \text{ arbitrary in } \mathbb{P}^1(\mathbb{C}). \tag{4.8}$$

(ii) $\mathscr{P}'(\partial\mathscr{H}^3)_- = \mathscr{P}'(\partial\mathscr{H}^3)/\{\{z\} + \{\bar{z}\} \mid z \in \mathbb{C} - \{0, 1\}\}$ *where* $\bar{z}$ *denotes the complex conjugate of* z. *Also there is a natural surjection* $\mathscr{P}'(\partial\mathscr{H}^3)_- \to \mathscr{P}(\partial\mathscr{H}^3)$ *with kernel of exponent dividing* 2. (*The kernel is* 0, *see* (5.24).)

Remark. In (4.8) any term involving a cross-ratio of non-distinct points is interpreted as zero. Thus at least 4 among $a_0, \ldots, a_4$ must be distinct. By Proposition 4.6, 3 of them can be taken to be $\infty, 0, 1$. (4.8) can then be written:

(4.9)(i) $\{z\} + \{z^{-1}\} = 0$;

(4.9)(ii) $\{z^{-1}\} - \{1 - z\} = 0$;

(4.9)(iii) $\{z_1\} - \{z_2\} + \{z_2/z_1\} - \{(1 - z_2)/(1 - z_1)\} + \{(1 - z_2)z_1/(1 - z_1)z_2\} = 0$

where $z, z_1, z_2 \in \mathbb{C} - \{0, 1\}$ and $z_1 \neq z_2$. Here (iii) corresponds to all a_i *distinct* and thus gives the defining relations for the group $\mathscr{P}_{\mathbb{C}}$ mentioned in the introduction. In the next section (Lemma 5.11) we shall see that (i) and (ii) are consequences of (iii) so that actually $\mathscr{P}_{\mathbb{C}} = \mathscr{P}'(\partial\mathscr{H}^3)$.

Next let $\mathbb{C}^\times$ be the multiplicative group of $\mathbb{C}$ and let $\Lambda^2_{\mathbb{Z}}(\mathbb{C}^\times)$ be the second exterior power written *additively* (i.e. it is the group of formal *sums* of symbols $a \wedge b$, $a, b \in \mathbb{C}^\times$; notice that $a \wedge b$ is bimultiplicative and $a \wedge a = 0$). Further let $\mu_{\mathbb{C}}$ ($\cong \mathbb{Q}/\mathbb{Z}$) be the group of roots of 1 in $\mathbb{C}^\times$. With this notation the theorem of Bloch–Wigner is:

Theorem 4.10. *There is an exact sequence* mod $\mathscr{C}_2$

$$0 \longrightarrow \mu_{\mathbb{C}} \longrightarrow H_3(\mathrm{PGL}(2, \mathbb{C}), \mathbb{Z}) \longrightarrow \mathscr{P}_{\mathbb{C}} \xrightarrow{\lambda} \Lambda^2_{\mathbb{Z}}(\mathbb{C}^\times) \xrightarrow{\mathrm{sym}} K_2(\mathbb{C}) \longrightarrow 0$$

where $\lambda\{z\} = z \wedge (1 - z)$ *and* $\mathrm{sym}(a, b) = \{a, b\}$ *is the* K_2*-symbol. Here* $K_2(\mathbb{C})$ *is* $H_2(\mathrm{SL}(2, \mathbb{C}), \mathbb{Z})$.

The first map in the above sequence is induced on H_3 by the natural inclusion of $\mu_{\mathbb{C}}$ into $\mathrm{PGL}(2, \mathbb{C})$. The second map is induced by the natural action of $\mathrm{PGL}(2, \mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$. In Appendix A, we have worked out the details of the proof of the theorem of Bloch–Wigner for any algebraically closed field F of characteristic 0 and we also treat the 2-torsion in the sequence. As it will be seen, $\varphi = 2\lambda$ is more appropriate than λ (-2λ is even better).

Remarks. 1. Proposition 4.5 for $\mathscr{P} = \mathscr{P}(\partial\mathscr{H}^3)$ follows from Theorem 4.10 by taking the (-1)-coeigenspace for complex conjugation in all terms. We note that complex conjugation induces the identity map on H_3 of $\mu_{\mathbb{C}}$ while $K_2(\mathbb{C})$ and $\Lambda^2_{\mathbb{Z}}(\mathbb{C}^\times)$ are actually $\mathbb{Q}$-vector spaces. We have also used (3.2) and Corollary 4.7. It is straightforward to check the commutativity of the following diagram:

$$\begin{array}{ccc} \mathscr{P}_{\mathbb{C}} & \xrightarrow{\varphi = 2\lambda} & \Lambda^2_{\mathbb{Z}}(\mathbb{C}^\times) \\ \downarrow & & \downarrow -\log^- \\ \mathscr{P}(\partial\mathscr{H}^3) & \xrightarrow{\Psi} & \mathbb{R}\otimes(\mathbb{R}/\mathbb{Z}) \end{array}$$

The left vertical map sends $\{z\}$ onto the formal asymptotic 3-simplex $(\infty, 0, 1, z)$. $\varphi\{z\} = 2.\ z\wedge(1-z)$ and Ψ is described by Lemma 4.4 using the canonical surjective map from $\mathscr{P}(\partial\mathscr{H}^3)$ to $\mathscr{P}(\mathscr{H}^3)$. In particular, $\{e^{2i\theta}\}$ is mapped onto $2\mathscr{L}(\theta)$ in $\mathscr{P}(\mathscr{H}^3)$ so that Ψ assigns to it the Dehn invariant:

$$2\log\sin\theta\otimes(2\theta/2\pi)\in\mathbb{R}\otimes(\mathbb{R}/\mathbb{Z})\cong\Lambda^2_{\mathbb{Z}}(\mathbb{C}^\times)^-.$$

On the other hand, $\varphi\{e^{2i\theta}\} = 2[e^{2i\theta}\wedge 2\sin\theta\, e^{-i(\pi-\theta)}]$. Note that $-\log^-$ is zero on $\Lambda^2_{\mathbb{Z}}(\mathbb{C}^\times)^+\cong\Lambda^2_{\mathbb{Z}}(\mathbb{R})\amalg\Lambda^2_{\mathbb{Z}}(\mathbb{R}/\mathbb{Z})$ and maps $r\wedge e^{2\pi i\alpha}$ onto $-\log|r|\otimes\alpha$. Aside from the appendix mentioned above, this calculation again shows that $\varphi = 2\lambda$ is more appropriate than λ.

2. The group $\mathscr{P}_{\mathbb{C},+} = \mathscr{P}_{\mathbb{C}}/\{\{z\} - \{\bar{z}\} \mid z\in\mathbb{C}-\{0,1\}\}$ appears to be related to the scissors congruence group $\mathscr{P}(S(\mathbb{R}^4)\}$. We shall investigate this elsewhere.

We end this section with some comments on the relation with the volume invariant $\mathrm{Vol}:\mathscr{P}(\mathscr{H}^3)\to\mathbb{R}$.

As mentioned in Sah [19; formula (4.9)], the volume of $\mathscr{L}(\theta)$, $0<\theta<\pi/2$ is given by

$$\mathrm{Vol}(\mathscr{L}(\theta)) = -\int_0^\theta \log 2\sin t\, dt = D(e^{2i\theta})/2$$

where $D:\mathbb{C}-\{0,1\}\to\mathbb{R}$ is the dilogarithm function defined by Bloch–Wigner (see Bloch [2; Section 6]):

$$D(z) = \arg(1-z)\log|z| - \mathrm{Im}\left\{\int_0^z \log(1-z)\, d(\log z)\right\}.$$

It then follows from (4.2) that the asymptotic simplex $(\infty, 0, 1, z)$ has volume

$$\mathrm{Vol}(\infty, 0, 1, z) = \{D(z/\bar{z}) + D((1-\bar{z})/(1-z)) + D((1-z)\bar{z}/(1-\bar{z})z)\}/2. \tag{4.11}$$

Now one can prove (cf. Bloch [2; Section 6 and Lemma 7.4.4]) that D satisfies

(4.12)(i) $D(z) + D(\bar{z}) = 0$,

(4.12)(ii) $D(z) + D(z^{-1}) = 0$,

(4.12)(iii) $D(z) + D(1-z) = 0$,

(4.12)(iv) $D(z_1) - D(z_2) + D(z_2/z_1) - D((1-z_2)/(1-z_1)) + D((1-z_2)z_1/(1-z_1)z_2) = 0$, $z_1\neq z_2\in\mathbb{C}-\{0,1\}$.

It follows

$$\mathrm{Vol}(\infty, 0, 1, z) = \{D(z) - D(\bar{z})\}/2 = D(z). \tag{4.13}$$

Notice that (iv) of (4.12) is just the application of D to the defining relation (4.9) of $\mathscr{P}_{\mathbb{C}}$.

In Bloch [2; Section 6] D occurs as the 'imaginary part' of a more general function ε with values in $\mathbb{C}^{\times} \otimes_{\mathbb{Z}} \mathbb{C}$. We shall use the following slight modification: For $z \in \mathbb{C} - \{0, 1\}$ let $\varrho(z) \in \Lambda^2_{\mathbb{Z}}(\mathbb{C})$ be defined by

$$\varrho(z) = ((\log z)/2\pi\iota) \wedge ((\log(1-z))/2\pi\iota) + 1 \wedge (2\pi\iota)^{-2} \int_0^z \left\{ \frac{\log(1-t)}{t} + \frac{\log t}{1-t} \right\} \mathrm{d}t. \tag{4.14}$$

This expression is to be interpreted as follows (cf. Bloch [2; Section 6]):

For two arcs γ, γ' in $\mathbb{C}$ let us use the notation $\gamma * \gamma'$ for the arc γ followed by γ' (assuming compatibility of ends). Now let γ_0 be the arc $[0, 1/2]$ from 0 to $1/2$ and let γ be any arc in $\mathbb{C} - \{0, 1\}$ from $1/2$ to z. Then in (4.14) $\log t$ and $\log(1-t)$ are branches of the logarithm along $\gamma_0 * \gamma$ and $\gamma_0 * (1-\gamma)$ respectively and

$$\int_0^z \omega = \int_0^{1/2} \omega + \int_{\gamma} \omega = -\pi^2/6 + \int_{\gamma} \omega,$$

$$\omega = \left\{ \frac{\log(1-t)}{t} + \frac{\log t}{1-t} \right\} \mathrm{d}t.$$

With this interpretation $\varrho(z)$ is independent of the choice of γ. The second term of (4.14) is known as Roger's L-function (see Roger [16]). Notice that the integrand $\omega = \log(1-t)\,\mathrm{d}(\log t) - \log t\,\mathrm{d}(\log(1-t))$ is formally the 'analytic analogue' of the first term of (4.14). Now λ satisfies the formal relation corresponding to (4.9): For $z_1 \neq z_2 \in \mathbb{C} - \{0, 1\}$,

$$\lambda\{z_1\} - \lambda\{z_2\} + \lambda\{z_2/z_1\} - \lambda\{(1-z_2)/(1-z_1)\} + \lambda\{(1-z_2)z_1/(1-z_1)z_2\} = 0 \tag{4.15}$$

(this relation is implicit in Theorem 4.10, but can be proved easily as in Section 5 below). It is therefore natural to expect that: For $z_1 \neq z_2 \in \mathbb{C} - \{0, 1\}$,

$$\varrho\{z_1\} - \varrho\{z_2\} + \varrho\{z_2/z_1\} - \varrho\{(1-z_2)/(1-z_1)\} + \varrho\{(1-z_2)z_1/(1-z_1)z_2\} = 0. \tag{4.16}$$

This in fact follows from the 'rigidity argument' of Bloch [2; Section 6, Lemma 6.2.2] in the same way as the proof of his Lemma 7.4.4. Thus ϱ induces an additive homomorphism:

$$\varrho : \mathscr{P}_{\mathbb{C}} \to \Lambda^2_{\mathbb{Z}}(\mathbb{C}).$$

If we let $e : \Lambda^2_{\mathbb{Z}}(\mathbb{C}) \to \Lambda^2_{\mathbb{Z}}(\mathbb{C}^{\times})$ be the exponential

$$e(z \wedge w) = \exp(2\pi\iota z) \wedge \exp(2\pi\iota w), \quad z, w \in \mathbb{C},$$

then Theorem 4.10 (actually the more precise version given in Appendix A) readily gives:

Proposition 4.17. *There is a commutative row exact diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_3(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z})/\mu_{\mathbb{C}} & \longrightarrow & \mathscr{P}_{\mathbb{C}} & \xrightarrow{\lambda} & \Lambda^2_{\mathbb{Z}}(\mathbb{C}^\times) \\ & & \downarrow c & & \downarrow \varrho & & \downarrow \\ 0 & \longrightarrow & \mathbb{C}/\mathbb{Q} & \xrightarrow{1\wedge \mathrm{id}} & \Lambda^2_{\mathbb{Z}}(\mathbb{C}) & \xrightarrow{e} & \Lambda^2_{\mathbb{Z}}(\mathbb{C}^\times) \end{array}$$

Remark. Notice that the natural map

$$\iota \cdot \mathrm{Im} : \Lambda^2_{\mathbb{Z}}(\mathbb{C}) \longrightarrow \Lambda^2_{\mathbb{Z}}(\mathbb{C})_- = \mathbb{R} \otimes_{\mathbb{Z}} \iota\mathbb{R} \xrightarrow{\text{mult.}} \iota\mathbb{R}$$

splits the inclusion

$$1 \wedge \mathrm{id} : \iota\mathbb{R} \longrightarrow \Lambda^2_{\mathbb{Z}}(\mathbb{C}).$$

With this notation it follows that

$$\begin{aligned} \mathrm{Im}\, \varrho\{z\} &= -(2\pi\iota)^{-2}\{D(z) - D(1-z)\} \\ &= D(z)/2\pi^2 = \mathrm{Vol}(\infty, 0, 1, z)/\mathrm{Vol}_3(S(\mathbb{R}^4)). \end{aligned} \tag{4.18}$$

Here Vol_3 is the 3-dimensional 'surface area' of the unit 4-ball. Hence by Dupont [8; Section 6, Remark 3] the map c in Proposition 4.17 is the evaluation of the Cheeger–Simons class $\hat{C}_2$ at least on $H_3(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z})_-$. The corresponding statement for $H_3(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z})_+$ is related to Remark 2 following Theorem 4.10.

5. Divisibility of $\mathscr{P}_F$ when F is algebraically closed

We shall now study $\mathscr{P}_F$ more closely and in particular we shall prove:

Theorem 5.1. *If F is an algebraically closed field of any characteristic, then $\mathscr{P}_F$ is divisible. If F is a real closed field, then $\mathscr{P}_F$ is 2-divisible.*

Corollary 5.2. (i) *$\mathscr{P}(\partial\mathscr{H}^3)$ and hence also $\mathscr{P}(\mathscr{H}^3)$ and $\mathscr{P}(\mathscr{H}^3)$ are divisible.*

(ii) *$H_3(\mathrm{SL}(2,F),\mathbb{Z})$ is divisible when F is any algebraically closed field of characteristic* 0.

Proof. (i) follows from Theorem 5.1 in view of Theorem 2.1, Proposition 3.7 and Corollary 4.7.

(ii) follows from Theorem 5.1 in view of Theorem 4.10 or rather the more precise version in Appendix A. □

It remains for us to prove Theorem 5.1. We will actually prove more. First note that similar to Corollary 4.7, $\mathscr{P}_F$ is generated by $\{z\} = [(\infty, 0, 1, z)]$, $z \in F - \{0, 1\}$,

subject to the relations

$$\{z_1\} - \{z_2\} + \{z_2/z_1\} - \{(1-z_2)/(1-z_1)\} + \{(1-z_2)z_1/(1-z_1)z_2\} = 0$$
$$\text{where } z_1 \neq z_2 \in F - \{0,1\}. \tag{5.3}$$

Let us deduce a few easy consequences of (5.3):

Lemma 5.4. *For $z \in F - \{0,1\}$, the following hold:*

(i) $2[\{z\} + \{z^{-1}\}] = 0$.

(ii) $\{z^2\} + \{z^{-2}\} = 0$.

(iii) $\{z\} + \{z^{-1}\} = \{-z\} + \{-z^{-1}\} + 2\{-1\}$ *when* $z \neq 0, \pm 1$ *and F has characteristic not 2; in particular,* $4\{-1\} = 0$.

Proof. The last term of (5.3) can be written as $\{(1-z_2^{-1})/(1-z_1^{-1})\}$. If we replace z_j by z_j^{-1} in (5.3) and add the resulting equations, we have

$$\{z_2/z_1\} + \{z_1/z_2\} = [\{z_2\} + \{z_1^{-1}\}] - [\{z_1\} + \{z_1^{-1}\}]. \tag{5.5}$$

Setting $z = z_2/z_1$ and using skew symmetry on the right hand side of (5.5) relative to the exchange of z_1 and z_2 we get (i). Putting $z_j = z^j$, $j = 1,2$, in (5.5) and using (i) we get (ii). Putting $z_2 = z = -z_1$ in (5.5) gives (iii). □

Lemma 5.6. *For $z_1, z_2 \in F - \{0,1\}$, the following hold:*

(i) $2[\{z_1\} + \{1-z_1\}] = 2[\{z_2\} + \{1-z_2\}]$.

(ii) $\{z_1\} + \{1-z_1\} = \{z_2\} + \{1-z_2\}$ *if* $z_1(1-z_1)z_2(1-z_2) \in F^{\times 2}$.

(iii) $6[\{z_1\} + \{1-z_1\}] = 0$ *if F has characteristic not 2.*

Proof. Replacing z_j by $1-z_j$ in (5.3) and adding the results, we have:

$$[\{z_2\} + \{1-z_2\}] - [\{z_1\} + \{1-z_1\}] = \{(1-z_1^{-1})/(1-z_2^{-1})\} + \{(1-z_2^{-1})/(1-z_1^{-1})\}. \tag{5.7}$$

(i) and (ii) now follow respectively from (i) and (ii) of Lemma 5.4. For (iii), take $z_2 = 1/2$ and $z_1 = 2$ in (i). From (i) and (iii) of Lemma 5.4, we see that $6\{1/2\} = 2\{-1\}$ so that $12\{1/2\} = 0$. (iii) therefore follows from (i) by taking $z_2 = 1/2$ and multiplying the result by 3. □

Lemma 5.8. *Assume F has characteristic not 2 and $z \in F - \{0, \pm 1\}$. The following hold:*

(i) $\{z^2\} = 2\{z\} + 2\{-z\} + 2\{-1\}$.

(ii) $\{-z^2\} + \{1+z^2\} = 2\{1/2\} + \{-z^2/(1+z^2)\} + \{-(1+z^2)/z^2\}$.

(iii) $\mathscr{P}_F$ *is 2-divisible when F is real closed.*

Proof. Taking $z_j = z^j$, $j = 1,2$, in (5.3), we get:

$$\{z\} - \{z^2\} + \{z\} - \{1+z\} + \{1+z^{-1}\} = 0. \tag{5.9}$$

Taking $z_1 = -z$ and $z_2 = -z^{-1}$ in (5.7), we get

$$\{-z^{-1}\} + \{1+z^{-1}\} - \{-z\} - \{1+z\} = \{z^{-1}\} + \{z\}. \tag{5.10}$$

(i) follows from putting together (5.9), (5.10) with (iii) of Lemma 5.4. Taking $z_1 = 1/2$ and $z_2 = -z^2$ in (5.7)) we get (ii). In any real closed field, $1+z^2$ is always a square and every nonzero element is either a square or the negative of a square. The right hand side of (ii) is $2\{1/2\} + 2\{-1\}$ by using (iii) of Lemma 5.4. (iii) therefore follows from (i) and (ii). □

Lemma 5.11. *Assume $F^\times = F^{\times 2}$. The following hold for $z \in F - \{0,1\}$:*

(i) $\{z\} + \{z^{-1}\} = 0$.

(ii) $\{z\} + \{1-z\} = 0$ *provided that $X^2 - X + 1 = 0$ has a solution in F.*

Proof. (i) follows from (ii) of Lemma 5.4. Using (ii) of Lemma 5.6, it is enough to verify (ii) for a single z in $F - \{0,1\}$. If $z^2 - z + 1 = 0$ then (ii) follows from (i). We note that the provision in (ii) is automatically satisfied when F has characteristic not 2 through the hypothesis $F^\times = F^{\times 2}$. □

From now on F is assumed to be an algebraically closed field. Using Lemma 5.11, we can extend the definition of $\{z\} \in \mathscr{P}_F$ allowing $\{z\}$ with $z \in \mathbb{P}^1(F) = F \cup \{\infty\}$ and dropping restrictions on z_1 and z_2 in (5.3). It is easy to see that we have done nothing more than setting $\{\infty\} = \{0\} = \{1\} = 0$ and interpreting ∞ in the usual manner. Equivalently, we allow for all 4-tuples (a_0, a_1, a_2, a_3), $a_i \in \mathbb{P}^1(F)$, as generators and set $(a_0, a_1, a_2, a_3) = 0$ whenever $a_i = a_j$ for some $i \neq j$ (cf. Remark after Corollary 4.7).

Following Bloch [2; Section 5] we define for two rational functions $f, g \in F(t)$ the $*$-product $f^- * g \in \mathscr{P}_F$ as follows:

Let $f(t) = a\prod(\alpha_i - t)^{d(i)}$, $d(i) \in \mathbb{Z}$, α_i distinct, $g(t) = b\prod(\beta_j - t)^{e(j)}$, $e(j) \in \mathbb{Z}$, β_j distinct. Put

$$f^- * g = \sum'_{i,j} d(i)e(j)\{\alpha_i^{-1}\beta_j\}, \quad \text{the sum extends over } i,j \text{ with } \alpha_i, \beta_j \in F^\times \text{ and the expression is 0 if } f \text{ or } g \in F. \tag{5.12}$$

It is immediate that $f^- * g$ is bimultiplicative (or rather 'bilogarithmic') on $F(t)^\times \times F(t)^\times$ and in view of (i) of Lemma 5.11 it is alternating:

$$f^- * f = 0 \quad \text{for all } f \in F(t)^\times. \tag{5.13}$$

We can now formulate an interesting identity in $\mathscr{P}_F$:

Theorem 5.14. *Let F be an algebraically closed field and let $f \in F(t)$. The following holds in $\mathscr{P}_F$:*

$$f^- * (1-f) = \{f(0)\} - \{f(\infty)\}.$$

Before proving this theorem let us first note that it generalizes the defining relation (5.3):

Lemma 5.15. *Suppose that* $f(t)=c(\alpha-t)/(\beta-t)\in F(t)^\times$, $\alpha,\beta,c\in F$ *and* $\alpha\neq\beta$. *Then*

$$f^{-}*(1-f)=\{f(0)\}-\{f(\infty)\}.$$

Proof. We may assume that either α or $\beta\neq 0$. For $c=1$,

$$1-f(t)=(\beta-\alpha)/(\beta-t)$$

and the desired identity becomes a consequence of (i) of Lemma 5.11:

$$\{1\}-\{\alpha^{-1}\beta\}=\{\alpha\beta^{-1}\}-\{1\}.$$

For $c\neq 1$,

$$1-f(t)=(1-c)(\gamma-t)/(\beta-t)\quad\text{with }\gamma=(\beta-c\alpha)/(1-c)$$

and the desired identity is

$$\{\gamma/\alpha\}-\{\gamma/\beta\}-\{\beta/\alpha\}=\{c\alpha/\beta\}-\{c\}. \tag{5.16}$$

This is equivalent to (5.3) with $z_1=c$, $z_2=c\alpha/\beta$. We note that 'degenerate' cases have been taken into account by the extension of (5.3). □

We next give a direct proof of (4.15). In view of (5.16), this is equivalent to the following:

Lemma 5.17. *Let* α,β,γ *be distinct in* $F^\times$ *and* $c,d\in F^\times$ *so that*

$$f(t)=c(\alpha-t)/(\beta-t)\quad\textit{and}\quad 1-f(t)=d(\gamma-t)/(\beta-t).$$

Then the following holds in $\Lambda^2_{\mathbb{Z}}(F^\times)$:

$$\lambda\{\gamma/\alpha\}-\lambda\{\gamma/\beta\}-\lambda\{\beta/\alpha\}=\lambda\{f(0)\}-\lambda\{f(\infty)\}.$$

Proof. First notice the following identities:

$$\begin{aligned}&1=1-f(\alpha)=d(\gamma-\alpha)/(\beta-\alpha),\\ &1=f(\gamma)=c(\alpha-\gamma)/(\beta-\gamma),\\ &(\gamma-\beta)/(\alpha-\beta)=c/d.\end{aligned} \tag{5.18}$$

Next recall that $\lambda\{z\}=z\wedge(1-z)$ and $\varphi=2\lambda$ so that

$$\begin{aligned}&\lambda\{\gamma/\alpha\}=(\gamma/\alpha)\wedge(\alpha-\gamma)/\alpha=\gamma\wedge(\alpha-\gamma)-\alpha\wedge(\alpha-\gamma)-\gamma\wedge\alpha,\\ &-\lambda\{\gamma/\beta\}=-\gamma\wedge(\beta-\gamma)+\beta\wedge(\beta-\gamma)+\gamma\wedge\beta,\\ &-\lambda\{\beta/\alpha\}=-\beta\wedge(\alpha-\beta)+\alpha\wedge(\alpha-\beta)+\beta\wedge\alpha.\end{aligned}$$

(N.B. we are working with $\Lambda^2_{\mathbb{Z}}(F^\times)$, not $\Lambda^2_{\mathbb{Z}}(F)$.) Using (5.18) we obtain

$$\begin{aligned}\lambda\{\gamma/\alpha\}-\lambda\{\gamma/\beta\}-\lambda\{\beta/\alpha\} &= -\gamma\wedge(c\alpha/\beta)+\beta\wedge(-c\alpha/d)+\alpha\wedge d\\ &= \lambda\{f(0)\}-[(\beta/d)\wedge(c\alpha/\beta)-\beta\wedge(c\alpha)+\beta\wedge d-\alpha\wedge d]\\ &= \lambda\{f(0)\}-c\wedge d=\lambda\{f(0)\}-\lambda\{f(\infty)\}.\end{aligned}$$

Note that $\beta\wedge(-1)=0$ holds in $\Lambda^2_{\mathbb{Z}}(F^\times)$ because $F^\times=F^{\times 2}$. □

Proof of Theorem 5.14. For $f\in F(t)$ put

$$L(f)=f^-*(1-f)\in\mathscr{P}_F \quad\text{and}\quad R(f)=\{f(0)\}-\{f(\infty)\}\in\mathscr{P}_F.$$

Using (5.13) and Lemma 5.11 it is easily seen that

$$\begin{aligned}&L(f)+L(f^{-1})=0=R(f)+R(f^{-1}), \qquad f\in F(t)^\times,\\ &L(f)+L(1-f)=0=R(f)+R(1-f), \quad f\in F(t).\end{aligned} \tag{5.19}$$

Next observe that the proof of Lemma 5.17 is purely formal and uses only that $x\wedge y$ is bimultiplicative, alternating and $y\wedge(-1)=0$. Clearly f^-*g for $f,g\in F(t)$ has the same properties so that

$$\begin{aligned}&L(f_1)-L(f_2)+L(f_2/f_1)-L((1-f_2)/(1-f_1))\\ &\quad+L((1-f_2)f_1/(1-f_1)f_2)=0.\end{aligned} \tag{5.20}$$

Initially, we need to assume that $f_1\neq f_2\in F(t)-\{0,1\}$, however with the extended definitions, this restriction is easily seen to be unnecessary. The analogous equations with L replaced by R follow directly from the extended form of (5.3).

Now let $H=\{f\in F(t)\mid L(f)=R(f) \text{ in } \mathscr{P}_F\}$ so that Lemma 5.15, (5.19) and (5.20) imply that H is a subset of $F(t)$ with the following properties:

(5.21)(i) $(at+b)/(ct+d)\in H$ for all $a,b,c,d\in F$ with $ct+d\neq 0$.

(5.21)(ii) $f\in H$ implies that $1-f\in H$ and that $f^{-1}\in H$ when $f\neq 0$.

(5.21)(iii) If $f_1, f_2, f_2/f_1$ and $(1-f_2)/(1-f_1)\in H$, then $(1-f_2)f_1/(1-f_1)f_2\in H$.

However, the following lemma was proved for us by E. Thue Poulsen (see Dupont–Poulsen [9]):

Lemma 5.22. *If F is algebraically closed and $H\subset F(t)$ satisfies* (5.21), *then $H=F(t)$.*

Theorem 5.14 therefore follows. □

Theorem 5.23. *Let F be an algebraically closed field of characteristic $p\geq 0$. For $n\in\mathbb{N}$ let n_0 denote the largest factor of n prime to p for $p>0$ and $n_0=n$ for $p=0$. Let ξ be a primitive n_0-th root of* 1 *in F. The following 'distribution relations' hold in $\mathscr{P}_F$:*

$$\{z^n\}=n\sum_{0\leq j\leq n-1}\{\xi^j z\}, \quad z\in F.$$

Proof. Since $\{0\}=0$, we may take $z\neq 0$. The general case follows easily from the cases where $n=p>0$ is the characteristic of F and where n is prime to the characteristic of F. We consider them simultaneously. If $z^n=1$, then the left side is $\{1\}=0$ and we may assume that n is prime to the characteristic of F. Using (i) of Lemma 5.11, the right hand side is $\{1\}$ or $\{1\}+\{-1\}$ according to n is odd or even. For n even, (i) of Lemma 5.8 shows that $\{-1\}=2\{\imath\}+2\{-\imath\}+2\{-1\}$ with $\imath^2=-1$. Thus $\{-1\}=0$ follows from (i) of Lemma 5.11. We may now assume $z^n\in F-\{0,1\}$. Consider

$$f(t)=(1-t^n)/(1-z^n)=\left(\prod_{0\leq j\leq n-1}(\xi^j-t)\right)/(1-z^n),$$

$$1-f(t)=\left(\prod_{0\leq j\leq n-1}(\xi^j z-t)\right)/(z^n-1).$$

In either case,

$$f^{-}*(1-f)=n\sum_{0\leq j\leq n-1}\{\xi^j z\}.$$

On the other hand, we obtain from Lemma 5.11 and the extended definition

$$\{f(0)\}-\{f(\infty)\}=\{(1-z^n)^{-1}\}-\{\infty\}=\{z^n\}.$$

Theorem 5.23 therefore follows from Theorem 5.14. □

Proof of Theorem 5.1. The first assertion follows from Theorem 5.23 because $F^\times=F^{\times n}$. The second assertion is just (iii) of Lemma 5.8. □

Remarks. 1. We now improve (ii) of Corollary 4.7, i.e.

$$\mathscr{P}_{\mathbb{C},-}=\mathscr{P}'(\partial\mathscr{H}^3)_-\cong\mathscr{P}(\partial\mathscr{H}^3). \tag{5.24}$$

In fact, the natural surjection in (ii) of Corollary 4.7 has kernel generated by the images of $\{r\}$ in $\mathscr{P}_{\mathbb{C},-}$ with $r\in\mathbb{R}$. With $n=2$ in Theorem 5.23, we have

$$\{s^2\}=2[\{s\}+\{-s\}] \quad \text{in } \mathscr{P}_{\mathbb{C}},\ s\in\mathbb{R};$$

and

$$\{-s^2\}=2[\{\imath s\}+\{-\imath s\}] \quad \text{in } \mathscr{P}_{\mathbb{C}},\ s\in\mathbb{R}.$$

Since $\bar{s}=s$ and $\overline{\imath s}=-\imath s$ hold for $s\in\mathbb{R}$, the above equations imply

$$\{\{r\}\mid r\in\mathbb{R}\}\subset\{\{z\}+\{\bar{z}\}\mid z\in\mathbb{C}\} \quad \text{as subgroups of } \mathscr{P}_{\mathbb{C}}. \tag{5.25}$$

It follows that (5.24) holds. We note in addition that the inclusion in (5.25) is strict. To see this we use $\lambda:\mathscr{P}_{\mathbb{C}}\to\Lambda^2_{\mathbb{C}}(\mathbb{C}^\times)$. Notice that $\{z\}+\{\bar{z}\}$ lies in $\mathscr{P}^+_{\mathbb{C}}$ so that λ carries it into $\Lambda^2_{\mathbb{Z}}(\mathbb{C}^\times)^+\cong\Lambda^2_{\mathbb{Z}}(\mathbb{R})\amalg\Lambda^2_{\mathbb{Z}}(\mathbb{R}/\mathbb{Z})$. If $z=ae^{\imath\alpha}$ and $1-z=be^{\imath\beta}$, then the component of $z\wedge(1-z)$ in $\Lambda^2_{\mathbb{C}}(\mathbb{C}^\times)^+$ is

$$\log a\wedge\log b+(\beta/2\pi)\wedge(\alpha/2\pi). \tag{5.26}$$

We note that the first term lies in $\Lambda^2_{\mathbb{Z}}(\mathbb{R})$. In view of the fact that $(\beta/2\pi)\wedge(\alpha/2\pi)$ is unchanged if $\alpha/2\pi$ is modified by the addition of a rational multiple of $\beta/2\pi$ (and vice versa) it is clear that the last term ranges over a set of generators of $\Lambda^2_{\mathbb{Z}}(\mathbb{R}/\mathbb{Z})$. In a similar manner, it is easy to see that the first term also ranges over a set of generators of $\Lambda^2_{\mathbb{Z}}(\mathbb{R})$. If $z=r\in\mathbb{R}$, then the last term is 0 while the first becomes $\log|r|\wedge\log|1-r|$, $r\in\mathbb{R}-\{0,1\}$. This clearly shows the strict inclusion in (5.25). Indeed, $\mathbb{R}$ can be replaced by any real closed field and $\mathbb{C}$ then denote its algebraic closure. In passing, we also note that:

$$\mathscr{P}^+_{\mathbb{C}}=\{\{z\}+\{\bar{z}\}\mid z\in\mathbb{C}\}\ \amalg \text{ a group of exponent dividing 2.} \tag{5.27}$$

The unknown group of exponent dividing 2 is 0 if and only if $\mathscr{P}_{\mathbb{C},-}$ is free of 2-torsion (i.e. uniquely 2-divisible). Note that $\mathscr{P}_{\mathbb{C},-}$ is generated by $\{e^{i\theta}\}$, $\theta\in\mathbb{R}$. As shown before $\{-1\}=0$ in $\mathscr{P}_{\mathbb{C}}$ so the trivial candidate for 2-torsion is actually 0. On the other hand, Theorem 5.23 furnishes many candidates for 2-torsion. Namely, let $u,v\in\mathbb{C}-\{0,1\}$ so that $u^2+v^2=1$ Lemma 5.11 together with Theorem 5.23 imply that

$$2[\{u\}+\{-u\}+\{v\}+\{-v\}]=0 \quad \text{in } \mathscr{P}_{\mathbb{C}}.$$

It is not obvious that $\{u\}+\{-u\}+\{v\}+\{-v\}$ is 0 in $\mathscr{P}_{\mathbb{C}}$ or that it has image 0 in $\mathscr{P}_{\mathbb{C},-}$. Similar candidate exist for elements of order n with any $n>0$.

2. In [15], Milnor stated a conjecture concerning the values of the volume function in $\mathscr{H}^3$. To be precise, let $V(\theta)=\operatorname{vol}\mathscr{L}(\theta)=D\{e^{2i\theta}\}/2$ with $0<\theta<\pi/2$. Using Theorem 5.23, $V(\theta)=D\{e^{i\theta}\}+D\{-e^{i\theta}\}$. We can extend the definition of V to all of $\mathbb{R}$ through the functional equations

$$V(0)=0, \qquad V(-\theta)=-V(\theta), \qquad V(\theta+\pi)=V(\theta).$$

With this extended definition, Milnor's conjecture is:

5.28. Assume $\theta\in\mathbb{Q}\pi$. Then every $\mathbb{Q}$-linear relation among $V(\theta)$ is a consequence of the following relations:

$$V(-\theta)=-V(\theta), \qquad V(\theta+\pi)=V(\theta), \qquad V(n\theta)=n\sum_{0\le j\le n-1}V(\theta+j\pi/n).$$

Milnor also noted that conjecture 5.28 is equivalent to:

5.29. For any $n>2$, the following real numbers are $\mathbb{Q}$-linearly independent:

$$V(j\pi/n), \quad 0<j<n/2, \ (j,n)=1.$$

Evidently, 5.29 implies the following conjecture:

5.30. For any $n>2$, the following elements in $\mathscr{P}_{\mathbb{C}}$ are $\mathbb{Q}$-linearly independent mod torsion:

$$\{e^{2\pi ij/n}\}, \quad 0<j<n/2, \ (j,n)=1.$$

If $z \in \mu_{\mathbb{C}}$ (all roots of 1 in $\mathbb{C}$), then $\lambda\{z\} = 2[z \wedge (1-z)] = 0$ in $\Lambda^2_{\mathbb{Z}}(\mathbb{C}^\times)$ and $\{z\}$ lies in $\mathscr{P}^-_{\mathbb{C}}$. Recall that $\mathscr{P}^-_{\mathbb{C}}$ maps onto $\mathscr{P}_{\mathbb{C},-}$ with kernel of exponent dividing 2. If we now assume the conjecture that volume and Dehn invariant separate the points of $\mathscr{P}(\mathscr{H}^3) \cong \mathscr{P}(\check{\mathscr{H}}^3)$, then 5.30 and 5.29 are equivalent. Notice that the additional assumption forces $\mathscr{P}_{\mathbb{C}} \cong \mathscr{P}(\partial\mathscr{H}^3)$ to be a $\mathbb{Q}$-vector space. In fact, $\mathscr{P}(\partial\mathscr{H}^3)$ is forced to be isomorphic to $\mathscr{P}(\check{\mathscr{H}}^3)$ because $\mathscr{P}_{\mathbb{C}}$ is divisible and $\mathscr{P}(\partial\mathscr{H}^3)$ maps surjectively onto $\mathscr{P}(\check{\mathscr{H}}^3)$ with kernel of exponent dividing 2.

Furthermore, the fact that $\lambda\{z\} = 0$ for $z \in \mu_{\mathbb{C}}$ allows us to conclude from the Bloch–Wigner theorem that these $\{z\}$ represent elements of $H_3(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z})$ with indeterminacy lying in $\mathbb{Q}/\mathbb{Z} \cong \mu_{\mathbb{C}} \subset H_3(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z})$. Milnor's conjecture would therefore give an explicit proof of the known assertion that $H_3(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z})$ has infinite $\mathbb{Q}$-rank. For a non-explicit proof, see Cheeger [5].

In a private communication, Milnor pointed out that 5.28 is not valid if θ were allowed to range over $\mathbb{R}$ so that $e^{i\theta}$ is algebraic over $\mathbb{Q}$. To be specific, Milnor used an 'exotic' formula obtained by Lobatchevskii [13; p. 124 with $L(x) = x \log 2 - V((\pi/2) - x)$]:

$$\begin{aligned} &V(x) + V(x') + V(y) + V(y') - V(z) - V(z') \\ &\quad = (V(z+x-y) + V(z+y-x) - V(z-x-y) - V(z+x+y))/2, \qquad (5.31) \\ &x + x' = y + y' = z + z' = \pi/2 \text{ and } \tan z = \sin(x+y)/\cos(x-y). \end{aligned}$$

Taking $x = y = \pi/6$ so that $2\cos z = 4/7^{1/2}$ and $z \notin \mathbb{Q}\pi$ but e^{iz} is algebraic, (5.31) then yields

$$\begin{aligned} &2V(z) - V(z-(3\pi/6)) - (V(z-(2\pi/6)) \\ &\quad + V(z-(4\pi/6)))/2 \equiv 0 \mod \mathbb{Q} \cdot V(\pi/6). \end{aligned}$$

However, the extended conjecture would imply that $V(z + j\pi/n)$, $0 \le j \le n-1$, are $\mathbb{Q}$-linearly independent modulo the $\mathbb{Q}$-subspace generated by all $V(\theta)$ with $\theta \in \mathbb{Q}\pi$ as long as $z \notin \mathbb{Q}\pi$ and $n > 0$. For the particular z above and $n = 6$, this is not the case.

We note that (5.31) is actually valid on the level of $\mathscr{P}_{\mathbb{C}}$ after we multiply through by 2 and replace $V(\theta)$ by $\{e^{i\theta}\}$. The verification amounts to applying Theorem 5.14 to the following rational function in $\mathbb{C}(t)$:

$$f(t) = (e^{2iz} - t^2)/\cos z\,(1 + 2i \sin(x+y)\,t - t^2)$$

and using Theorem 5.23 for $n = 2$. We omit the details.

3. The discussions in the preceding two remarks are valid for the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. It is known that $K_2(\bar{\mathbb{Q}}) = 0$ so that $\Lambda^2_{\mathbb{Z}}(\bar{\mathbb{Q}}^\times)$ is generated by $z \wedge (1-z)$ with z ranging over $\bar{\mathbb{Q}} - \{0, 1\}$. However, an elementary proof of this fact does not seem to be available. The rigidity property of the Cheeger–Simons invariant suggests that perhaps the inclusion of $\bar{\mathbb{Q}}$ in $\mathbb{C}$ induces a surjective map from $H_3(\mathrm{SL}(2,\bar{\mathbb{Q}}),\mathbb{Z})$ to $H_3(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z})$.

4. Theorem 5.23 suggests that $\mathscr{P}_F$ should be *uniquely* divisible by n through the formula

$$\{w\}/n = \sum_{0 \le j \le n-1} \{\xi^j w^{1/n}\}, \quad \xi \text{ primitive } n\text{-th root of } 1.$$

However, it is not clear that this formula respects the defining relation (5.3). If F is algebraically closed of characteristic $p>0$, then $\{w\}/p = p\{w^{1/p}\}$ does respect (5.3) so that $\mathscr{P}_F$ is in fact uniquely p-divisible.

6. Spherical fundamental polytopes

In the spherical cases, we have the general isomorphism

$$\mathscr{P}(S(\mathbb{R}^{2i+1})) \cong \mathscr{P}(S(\mathbb{R}^{2i})), \quad i \le 0;$$

it is known (and easy) that $\mathscr{P}(S(\mathbb{R}^0)) \cong \mathbb{Z}$ and $\mathscr{P}(S(\mathbb{R}^2)) \cong \mathbb{R}$, see Sah [19; Theorem 2.6]. In Sah [18; Chapter 6], a Hopf algebra structure was introduced to summarize some basic geometric facts as well as to facilitate the further study of the structure of these scissors congruence groups. The classical Dehn invariant (in 3 dimensional spherical space) was modified and extended to all dimensions in order to obtain a comodule structure map. It appears reasonable to conjecture that $\mathscr{P}(S(\mathbb{R}^n))$ is torsionfree (and perhaps divisible when $n>1$). We note that the absence of torsion is equivalent with the conjecture that the (modified) classical Dehn invariants (including volume) should separate the points of the spherical scissors congruence groups, see Sah [18; Proposition 3.22, p. 118]. However, it should be noted that the modification already occurred in the definition of the classcical Dehn invariant in dimension 3. As a result, the absence of torsion in $\mathscr{P}(S(\mathbb{R}^4))$ is not yet known to be equivalent to the conjecture that volume and Dehn invariant separate the scissors congruence classes in 3 dimensional spherical space. In any case, the torsion subgroup of $\mathscr{P}(S(\mathbb{R}^{2i}))$, $i>1$, can be seen to be isomorphic to the torsion subgroup of $\mathscr{P}(S(\mathbb{R}^{2i}))$/suspension.

As a test, we study the subgroup of $\mathscr{P}(S(\mathbb{R}^n))$, $n \ge 2$, arising from fundamental domains of finite subgroups of $O(n, \mathbb{R})$. Our main result is that this contributes a direct summand of $\mathbb{Q}$ to $\mathscr{P}(S(\mathbb{R}^n))$. This resolves a question raised in Sah [18; p. 128]; namely, these fundamental domains do not lead to torsion and their scissors congruence classes are determined by volume alone. We note also that these are responsible for a direct summand of $\mathbb{Q}/\mathbb{Z}$ in $H_{2i-1}(\mathrm{SO}(2i, \mathbb{R}), \mathbb{Z})$. In case $n=4$, $i=2$, our present result is already implicit in Dupont [8; Corollary 5.36 and remarks]; by comparison, the present approach is more direct.

Proposition 6.1. *Let $S(\mathbb{R}^n)$ denote the sphere of all unit vectors in $\mathbb{R}^n$. For each finite subgroup G of $O(n, \mathbb{R})$, there is a fundamental polytype P for the action of G on $S(\mathbb{R}^n)$. If P' is another fundamental polytope for the action of G, then P and P' are G-scissors congruent.*

Proof. For the existence of P, we use the Dirichlet fundamental domain (also called the Poincaré fundamental domain). We recall its construction. For each $g \neq 1$ in G, the fixed points of g on $S(\mathbb{R}^n)$ lies on a proper subspace of $\mathbb{R}^n$. Since G is finite, we can find x in $S(\mathbb{R}^n)$ so that $gx \neq x$ for each $g \neq 1$ in G. The Dirichlet fundamental domain about x is

$$D(x) = \{ y \in S(\mathbb{R}^n) \mid \operatorname{dist}(y,x) \leq \operatorname{dist}(xy,x), g \in G \}.$$

The distance is understood to be the $O(n,\mathbb{R})$-invariant distance on the sphere. For fixed $g \neq 1$ in G, $\operatorname{dist}(y,x) \leq \operatorname{dist}(gy,x) = \operatorname{dist}(y, g^{-1}x)$ holds for $y \in S(\mathbb{R}^n)$ if and only if y lies on the hemisphere containing x determined by the hyperplane orthogonal to the vector $g^{-1}x - x$ in $\mathbb{R}^n$. As a consequence, $D(x)$ is the intersection of a finite number of hemispheres so that it is a convex spherical polytope P. Evidently,

$$g(D(x)) = D(gx) \quad \text{and} \quad S(\mathbb{R}^n) = \coprod_{g \in G} g(D(x)).$$

Suppose that P' is another fundamental polytope for the action of G on $S(\mathbb{R}^n)$. Then $D(x) = \coprod_{g \in G} gP' \cap D(x)$ and $P' = \coprod_{g \in G} P' \cap g^{-1}(D(x))$. Evidently,

$$gP' \cap D(x) = g(P' \cap g^{-1}(D(x)).$$

Since $gP' \cap D(x)$ either has empty interior or is a polytope, P' and $P = D(x)$ are G-scissors congruent. □

Remark. The preceding argument extends to Euclidean, hyperbolic or extended hyperbolic spaces as long as a fundamental polytope exists. This general result is due to Siegel [21; Lemma 3].

In view of Proposition 6.1, we can associate to each finite subgroup G of $O(n,\mathbb{R})$ a well defined element $[\mathrm{FD}(G)] \in \mathscr{P}(S(\mathbb{R}^n))$ where $\mathrm{FD}(G)$ is any fundamental polytope for the action of G on $S(\mathbb{R}^n)$. This element $[\mathrm{FD}(G)]$ depends only on the conjugacy class of G in $O(n,\mathbb{R})$.

In an abstract group K, two subgroups A and B are said to be directly cocommensurable if there exist elements x, y in K so that xAx^{-1} and yBy^{-1} are of finite index in some common subgroup C of K. Cocommensurability is then defined to be the equivalence relation on the set of subgroups of A generated by the relation of direct cocommensurability. For the case of $O(n;\mathbb{R})$, we will primarily be interested in the case of finite subgroups of the same common order, say M. Since $O(n,\mathbb{R})$ may contain maximal finite subgroups, it is not possible to show cocommensurability between arbitrary finite subgroups. In particular, cocommensurability is more restrictive than commensurability up to conjugacy.

Proposition 6.2. *Let A and B be finite subgroups of common order M in $O(n,\mathbb{R})$. Suppose that A and B are cocommensurable within the set of all subgroups of order M in $O(n,\mathbb{R})$. Then* $[\mathrm{FD}(A)] = [\mathrm{FD}(B)]$.

Proof. We may assume that A and B are directly cocommensurable. After conjugations, A and B can be assumed to be subgroups of the same index m in a suitable finite subgroup C of $O(n, \mathbb{R})$. FD(A) and FD(B) are therefore both interior disjoint unions of m translates under C of FD(C). Thus $[\mathrm{FD}(A)] = m[\mathrm{FD}(C)] = [\mathrm{FD}(B)]$. □

Proposition 6.3. *Let A and B be finite subgroups of common order M in $O(n, \mathbb{R})$. Suppose that for each prime p, a p-Sylow subgroup A_p of A is cocommensurable with a p-Sylow subgroup B_p of B within the set of all p-subgroups of $O(n, \mathbb{R})$ of order equal to $M_p = |A_p| = |B_p|$. Then* $[\mathrm{FD}(A)] = [\mathrm{FD}(B)]$.

Proof. We note: M/M_p is coprime to p. From Proposition 6.2,

$$(M/M_p)[\mathrm{FD}(A)] = [\mathrm{FD}(A_p)] = [\mathrm{FD}(B_p)] = (M/M_p)[\mathrm{FD}(B)].$$

$[\mathrm{FD}(A)] - [\mathrm{FD}(B)]$ therefore has order dividing the integers M/M_p for each prime p. Since M/M_p have greatest common divisor 1 as p ranges over primes, $[\mathrm{FD}(A)] = [\mathrm{FD}(B)]$ follows. □

Theorem 6.4. *Let A and B be two finite subgroups of the same order in $O(n, \mathbb{R})$. Then* $[\mathrm{FD}(A)] = [\mathrm{FD}(B)]$. *If $n \geq 2$, then the volume map induces an isomorphism between the subgroup of $\mathscr{P}(S(\mathbb{R}^n))$ generated by* $[\mathrm{FD}(A)]$ *with A ranging over all finite subgroups of $O(n, \mathbb{R})$ and the group $\mathbb{Q}$.*

Proof. Using Proposition 6.3, A and B can be taken to be p-subgroups of the same order and we show that they are directly cocommensurable. Finite p-groups have decreasing sequences of normal subgroups with successive factors of order p. By a theorem of Borel–Serre [3; Theorem 1], A and B can be conjugated into normalizers of maximal tori of $O(n, \mathbb{R})$. Since all maximal tori in $O(n, \mathbb{R})$ are conjugate, A and B can be taken to be contained in the normalizer N of a fixed maximal torus T. $W = N/T$ is the finite Weyl group associated to $O(n, \mathbb{R})$. Let T_0 be the torsion subgroup of T so that T_0 is a union of finite subgroups. Evidently T_0 is normal in N and we have the exact sequence

$$1 \longrightarrow T/T_0 \longrightarrow N/T_0 \longrightarrow W \longrightarrow 1. \tag{6.5}$$

Now T/T_0 is a $\mathbb{Q}$-vector space and W is a finite group so that $H^i(U, T/T_0)$ is 0 for $i > 0$ and any subgroup U of W. This means that (6.5) splits and any finite subgroup of N/T_0 can be conjugated into a fixed complement C/T_0 of T/T_0 in N/T_0. A and B can therefore be taken to be in C. We also have the exact sequence

$$1 \longrightarrow T_0 \longrightarrow C \longrightarrow W \longrightarrow 1. \tag{6.6}$$

Since W is finite and T_0 is a union of finite subgroups, C is also a union of finite subgroups. Thus A and B are contained in a suitable finite subgroup C_0 of C. In other words, A and B are directly cocommensurable. The desired equality now

follows from Proposition 6.2. The last assertion follows from the fact that $SO(2,\mathbb{R}) \cong \mathbb{R}/\mathbb{Z}$ contains finite cyclic groups of arbitrary order. □

Remark. In essence, the proof relies on the fact that an irreducible complex representation of a finite p-group is monomial. This has to be generalized a bit and can be accomplished by a proof analysis. The theorem of Borel–Serre extends this fact to arbitrary compact Lie groups. For a topologically minded reader, the result of Borel–Serre amounts to showing the existence of a fixed point of a finite p-group S on the coset space $X = G/N$. By a theorem of P.A. Smith

$$\chi(X) \equiv \chi(X^S) \mod p, \quad \chi \text{ denotes Euler characteristic,}$$

$\chi(G/N)$ is known to be 1, hence X^S is nonempty and S has a fixed point on G/N. Notice that for a finite set X, the theorem of Smith is just the classical counting lemma for finite p-groups. The fact that $\chi(G/N) = 1$ can be seen algebraically. By complexification, we may reduce ourselves to the case where G is a maximal compact subgroup K of a connected simple algebraic group $G_{\mathbb{C}}$ over $\mathbb{C}$. Using the Iwasawa decomposition of $G_{\mathbb{C}}$, it is immediate that K/T is homeomorphic to $G_{\mathbb{C}}/B$ for a Borel subgroup B of $G_{\mathbb{C}}$. We have the Bruhat decomposition: $G_{\mathbb{C}} = \coprod_{w \in W} BwB$ where W is the Weyl group of $G_{\mathbb{C}}$ and W and W can be identified with the Weyl group of K. $G_{\mathbb{C}}/B$ is therefore a cell complex with even-dimensional cells:

$$BwB/B \cong B_u/(wB_u w^{-1} \cap B_u), \quad B_u \text{ is the unipotent radical of } B.$$

We note that each BwB/B is homeomorphic to an affine space over $\mathbb{C}$. It follows that $\chi(G_{\mathbb{C}}/B) = \chi(K/T) = |W|$. Since K/T is a covering space of K/N with finite fiber $W = N/T$, $\chi(K/N) = 1$ follows.

For hyperbolic spaces, the statement corresponding to Theorem 6.4 is open. The basic idea of subdividing by means of smaller fundamental domains breaks down. Indeed, the theorem of Kazdan–Margoulis shows that the volume of fundamental domains is bounded from below. For Euclidean spaces, the corresponding statement was proven in Sah [18; Theorem 8.3.1, p. 168] by using the absence of torsion in $\mathscr{P}(\mathbb{R}^n)$.

Appendix A. A theorem of S. Bloch and D. Wigner

In this appendix we prove the theorem of Bloch–Wigner (Theorem 4.10) in a more precise version.

For any field F let μ_F denote the group of all roots of 1 in F. Recall that $\mathscr{P}_F$ is the abelian group with generators $\{z\}$, $z \in F - \{0,1\}$ and defining relations

$$\{z_1\} - \{z_2\} + \{z_2/z_1\} + \{(1-z_2)/(1-z_1)\} - \{(1-z_2)z_1/(1-z_1)z_2\} = 0$$
$$\text{where } z_1 \neq z_2 \in F - \{0,1\}. \tag{A1}$$

When F is algebraically closed, we may allow z to lie in $\mathbb{P}^1(F) = F \cup \{\infty\}$ and drop the restrictions on z_1, z_2 in (A1). This forces $\{\infty\} = \{0\} = \{1\} = 0$.

Theorem (Bloch–Wigner). *Let F be any algebraically closed field of characteristic 0. There is an exact sequence*

$$0 \longrightarrow \mu_F(2) \longrightarrow H_3(\mathrm{SL}(2,F),\mathbb{Z}) \longrightarrow \mathscr{P}_F \xrightarrow{\lambda} \Lambda^2_{\mathbb{Z}}(F^\times) \xrightarrow{\mathrm{sym}} K_2(F) \longrightarrow 0. \tag{A2}$$

$\mu_F(2)$ is just μ_F with $\mathrm{Aut}(F)$ *acting through the quadratic character and the first map is induced by the inclusion of μ_F into the diagonal of* $\mathrm{SL}(2,F)$. *For z in $F - \{0,1\}$, $\lambda\{z\} = z \wedge (1-z)$. For u, v in $F^\times$,* $\mathrm{sym}(u \wedge v) = \{u, v\} \in K_2(F) = H_2(\mathrm{SL}(2,F),\mathbb{Z})$ *denotes the K_2-symbol.*

The map from $H_3(\mathrm{SL}(2,F),\mathbb{Z})$ to $\mathscr{P}_F$ is induced by sending the homogeneous 3-simplex (g_0, g_1, g_2, g_3), $g_i \in \mathrm{SL}(2,F)$ onto $\{z\}$ with z denoting the cross-ratio $\{g_0(\infty) : g_1(\infty) : g_2(\infty) : g_3(\infty)\}$ of points $g_i(\infty) \in \mathbb{P}^1(F)$, $0 \le i \le 3$. Since this is not used in our discussions, we will not be concerned with its verification. We leave it to the reader to check that this map does not depend on the choice of ∞ as the base point. From now on F will be an algebraically closed field of characteristic 0 so that $\mu_F = \mathbb{Q}/\mathbb{Z}$.

Throughout this appendix $G = \mathrm{PSL}(2,F) = \mathrm{PGL}(2,F)$. This requires only the hypothesis that $F^\times = F^{\times 2}$. Let $\tilde{G} = \mathrm{SL}(2,F)$ so that we have the exact sequence of groups (valid for F of characteristic not 2)

$$1 \longrightarrow \{\pm I\} \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1. \tag{A3}$$

It is well known that G and $\tilde{G}$ are perfect groups (valid when F has at least 4 elements). Thus,

$$H_1(G,A) = H_1(\tilde{G},A) = 0 \quad \text{for any } G\text{-trivial module } A. \tag{A4}$$

Under the hypothesis $F^\times = F^{\times 2}$, $H_2(\tilde{G},\mathbb{Z}) \cong K_2(F)$, see Sah–Wagoner [20]. With F algebraically closed, $K_2(F)$ is known to be a $\mathbb{Q}$-vector space by a theorem of Bass–Tate [1]. The Hochschild–Serre spectral sequence attached to (A3) with coefficient in $\mathbb{Z}$ can be analyzed. We obtain:

(A5) $H_2(G,\mathbb{Z}) \cong H_2(\tilde{G},\mathbb{Z}) \amalg \mathbb{Z} \bmod 2$, $H_2(\tilde{G},\mathbb{Z}) \cong K_2(F)$ is a $\mathbb{Q}$-vector space and $\mathbb{Z} \bmod 2$ corresponds to the exact sequence (A3).

(A6) $H_3(G,\mathbb{Z})$ is a quotient group of $H_3(\tilde{G},\mathbb{Z})$ with kernel of order dividing 4.

Let G act on $\mathbb{P}^1(F)$ through the usual fractional linear transformation action. The stability subgroup of ∞ is then the Borel subgroup B formed by upper triangular matrices in G. The stability subgroup of ∞ and 0 is then the split torus $T \cong F^\times$ formed by the diagonal matrices in G. If we let $U \cong F^+$ denote the upper unipotent

matrices in G, then we have the split exact sequence

$$1 \longrightarrow U \cong F^+ \longrightarrow B \longrightarrow T \cong F^\times \longrightarrow 1. \tag{A7}$$

The action of T on U corresponds to multiplication action of $F^\times$ on F^+. For any subgroup S of G, let $\tilde{S}$ denote the inverse image of S in $\tilde{G}$ corresponding to (A3). We may let $\tilde{G}$ act on $\mathbb{P}^1(F)$ through G. B and T can then be replaced by $\tilde{B}$ and $\tilde{T}$. The exact sequence (A7) remains valid provided that we change the action of $F^\times$ on F^+ to multiplication after squaring. This amounts to the identification of the exact sequences

$$1 \longrightarrow \tilde{1} \longrightarrow \tilde{T} \longrightarrow T \longrightarrow 0, \tag{A8}$$

$$1 \longrightarrow \{\pm 1\} \longrightarrow F^\times \xrightarrow{2} F^\times \longrightarrow 1. \tag{A9}$$

We note that (A8) is just the restriction of (A3) to the split torus T of G. $T \cong F^\times$ is a divisible abelian group with torsion subgroup $\mu_F \cong \mathbb{Q}/\mathbb{Z}$. The homology of T can be computed rather easily:

$$H_*(T, \mathbb{Z}) = \begin{cases} \Lambda^*_{\mathbb{Z}}(F^\times/\mu_F), & * \text{ even}, \\ \Lambda^*_{\mathbb{Z}}(F^\times/\mu_F) \amalg (\mathbb{Q}/\mathbb{Z}), & * \text{ odd}. \end{cases} \tag{A10}$$

We note that $\Lambda^*_{\mathbb{Z}}(F^\times/\mu_F)$ is a $\mathbb{Q}$-vector space in positive degrees. The identification in (A10) is functorial in the sense that the m-th power map on T leads to multiplication by m^k on $\Lambda^k_{\mathbb{Z}}(F^\times/\mu_F)$ in its $\mathbb{Q}$-vector space structure. However, the $\mathbb{Q}/\mathbb{Z}$ part in (A10) arises through Bockstein so that there is a dimension shift. Thus the m-th power map on T, hence on μ_F, corresponds to multiplication by m^i on $\mathbb{Q}/\mathbb{Z}$ in degree $2i-1$.

The homology of B can be computed through the Hochschild–Serre spectral sequence associated to (A7). Since F is a $\mathbb{Q}$-vector space, we can use the 'center kills' lemma to conclude:

(A11) $H_*(B, \mathbb{Z}) \cong H_*(T, \mathbb{Z})$, the isomorphism is induced by the inclusion of T into B and the inverse is induced by the projection in (A7).

With these preliminary data out of our way, we begin the proof of the theorem of Bloch–Wigner following the ideas sketched by them.

Let C_k be the free abelian group with basis formed by all $(k+1)$-tuples of *distinct* points of the projective line $\mathbb{P}^1(F)$. For any prime p and any abelian group A, A/pA will often be written as A mod p. We allow for the possibility that $p=0$. With the usual simplicial boundary homomorphisms ∂, we have the exact sequence of G-modules (exact since $\mathbb{P}^1(F)$ is infinite)

$$\cdots \longrightarrow C_k \longrightarrow C_{k-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{A12}$$

G is exactly 3-transitive on $\mathbb{P}^1(F)$ so

$$C_k \text{ is } G\text{-free for } k \geq 2. \tag{A13}$$

In fact, we have the following identifications:

$$\begin{aligned} C_0 &= \mathbb{Z}G \otimes_{\mathbb{Z}B} \mathbb{Z}.(\infty) = \mathrm{ind}_B^G \mathbb{Z}.(\infty), \\ C_1 &= \mathbb{Z}G \otimes_{\mathbb{Z}T} \mathbb{Z}.(\infty, 0) = \mathrm{ind}_T^G \mathbb{Z}.(\infty, 0), \\ C_2 &= \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}.(\infty, 0, 1) = \mathrm{ind}_1^G \mathbb{Z}.(\infty, 0, 1). \end{aligned} \tag{A14}$$

In these identifications, $g \otimes (x_0, \ldots, x_k)$ is identified with $(gx_0, \ldots, gx_k)$, $x_j \in \mathbb{P}^1(F)$ and G acts on the first factor through left multiplication. We split up (A12) into the following three G-exact sequences:

$$0 \longrightarrow Z_0 \longrightarrow C_0 \longrightarrow \mathbb{Z} \longrightarrow 0, \tag{A15}$$

$$0 \longrightarrow Z_1 \longrightarrow C_1 \longrightarrow Z_0 \longrightarrow 0, \tag{A16}$$

$$\cdots \longrightarrow C_k \longrightarrow C_{k-1} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow Z_1 \longrightarrow 0. \tag{A17}$$

From (A15) and (A16) we obtain the long homology exact sequences

$$\cdots \longrightarrow H_k(G, Z_0) \longrightarrow H_k(G, C_0) \longrightarrow H_k(G, \mathbb{Z}) \longrightarrow H_{k-1}(G, Z_0) \longrightarrow \cdots, \tag{A18}$$

$$\cdots \longrightarrow H_k(G, Z_1) \longrightarrow H_k(G, C_1) \longrightarrow H_k(G, Z_0) \longrightarrow H_{k-1}(G, Z_1) \longrightarrow . \tag{A19}$$

So far, we can reduce all coefficients mod p. We can also replace $G, B, T, 1$ by $\tilde{G}, \tilde{B}, \tilde{T}, \tilde{1}$. We only lose the freeness assertion in (A13) for $\tilde{G}$.

Using $\tilde{G}$ in place of G, the projection map in (A3) then yields commutative exact ladders involving (A18) and (A19). These ladders can be analyzed in detail for suitable values of k. We first note that (A14) combines with Shapiro's lemma and (A11) to yield

$$\begin{aligned} &H_*(G, C_0) \cong H_*(T, \mathbb{Z}), \quad H_*(\tilde{G}, C_0) \cong H_*(\tilde{T}, \mathbb{Z}), \\ &H_*(G, C_1) \cong H_*(B, \mathbb{Z}) \cong H_*(T, \mathbb{Z}), \\ &H_*(\tilde{G}, C_1) \cong H_*(B, \mathbb{Z}) \cong H_*(\tilde{T}, \mathbb{Z}). \end{aligned} \tag{A20}$$

Using (A4), we obtain

$$\begin{aligned} &H_0(G, C_0) \cong H_0(\tilde{G}, C_0) \cong H_0(G, \mathbb{Z}) \cong H_0(\tilde{G}, \mathbb{Z}) \cong \mathbb{Z}, \\ &H_0(G, Z_0) = 0 = H_0(\tilde{G}, Z_0). \end{aligned} \tag{A21}$$

The map from $C_0 = \mathrm{ind}_B^G \mathbb{Z}.(\infty)$ to $\mathbb{Z}$ is just augmentation so that the induced homomorphism from $H_k(G, C_0)$ to $H_k(G, \mathbb{Z})$ in (A18) is simply the inclusion homomorphism from $H_k(T, \mathbb{Z})$ to $H_k(G, \mathbb{Z})$ after we identify through (A20). This inclusion homomorphism factors through the normalizer N of T in G. $W = N/T$ is the Weyl group of order 2 and inverts T. The discussion after (A10) applies with $m = -1$. As a result, for k odd, the $\mathbb{Q}$-vector space part $\Lambda_{\mathbb{Z}}^k(F^\times/\mu_F)$ of $H_k(T, \mathbb{Z})$ is mapped onto 0 in

$H_k(G,\mathbb{Z})$. Similarly, for $k \equiv 1 \bmod 4$, the $\mathbb{Q}/\mathbb{Z}$ part of $H_k(T,\mathbb{Z})$ is also mapped onto 0 in $H_k(G,\mathbb{Z})$. The same assertions hold for $\tilde{G}$, $\tilde{T}$ in place of G, T. According to the theorem of Matsumoto–Moore, the inclusion of the split torus $\tilde{T}$ into $\tilde{G} = \mathrm{SL}(2,F)$ induces surjective homomorphism on H_2, see Sah–Wagoner [2; Prop. 1.10, p. 617]. The ladder corresponding to (A18) may be terminated at $k=2$ with $H_2(\tilde{G},\mathbb{Z})$ and $H_2(G,\mathbb{Z})$ both replaced by $K_2(F)$. We also obtain the following commutative row exact diagram:

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & H_1(\tilde{G},Z_0) & \longrightarrow & F^\times & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow 2 & & \\ 0 & \longrightarrow & \mathbb{Z} \bmod 2 & \longrightarrow & H_1(G,Z_0) & \longrightarrow & F^\times & \longrightarrow & 0 \end{array} \tag{A22}$$

This shows:

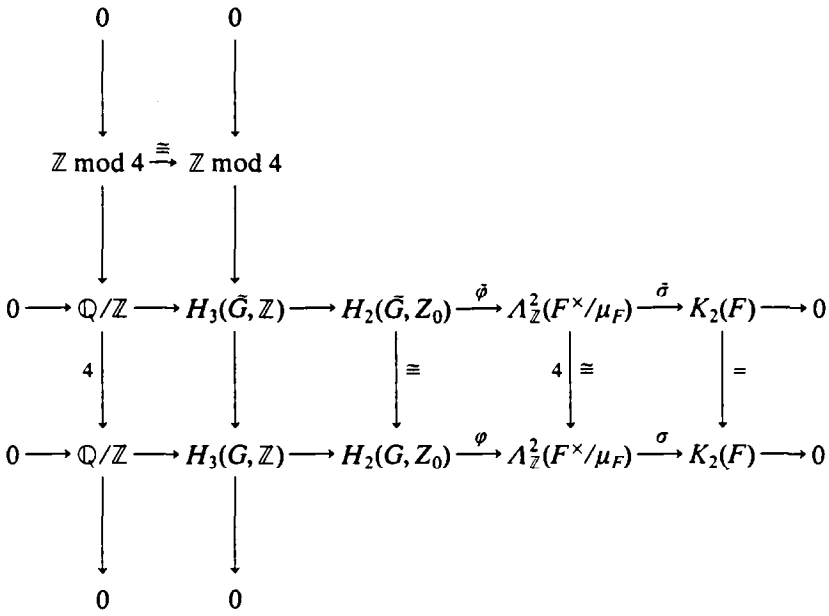
(A23) $H_1(\tilde{G},Z_0) \cong H_1(G,Z_0) \cong F^\times$ with the first isomorphism given through the projection map in (A3).

We note that the map from $H_1(\tilde{G},Z_0)$ to $H_1(G,Z_0)$ is surjective through the Hochschild–Serre spectral sequence associated to (A3) with coefficient in Z_0.

We can further extract from the ladder associated to (A18) the following commutative exact ladder:

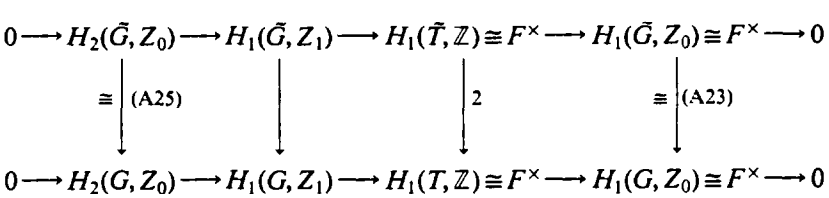
$$\begin{array}{ccccccccccc} \mathbb{Q}/\mathbb{Z} & \longrightarrow & H_3(\tilde{G},\mathbb{Z}) & \longrightarrow & H_2(\tilde{G},Z_0) & \longrightarrow & \Lambda^2_{\mathbb{Z}}(F^\times/\mu_F) & \longrightarrow & K_2(F) & \longrightarrow & 0 \\ \downarrow 4 & & \downarrow & & \downarrow & & \downarrow 4 & & \downarrow = & & \\ \mathbb{Q}/\mathbb{Z} & \longrightarrow & H_3(G,\mathbb{Z}) & \longrightarrow & H_2(G,Z_0) & \longrightarrow & \Lambda^2_{\mathbb{Z}}(F^\times/\mu_F) & \longrightarrow & K_2(F) & \longrightarrow & 0 \end{array} \tag{A24}$$

The map from $\mathbb{Q}/\mathbb{Z}$ to $H_3(\mathrm{SL}(2,F),\mathbb{Z})$ is induced by the inclusion of μ_F into the diagonal of $\mathrm{SL}(2,F)$. This map is actually injective. To see this, we note that homology is of finite character. Thus injectivity can be tested with μ_F replaced by a finite cyclic subgroup and F can be replaced by a finitely generated subfield. Since F is algebraically closed of characteristic 0, we can now replace the discrete group $\mathrm{SL}(2,F)$ by $\mathrm{SL}(2,\mathbb{C})$ in testing injectivity (a sort of 'Lefshetz Principle'). With F taken to be $\mathbb{C}$, the finite cyclic groups are then mapped into $\mathrm{SU}(2,\mathbb{C})$. The injectivity can now be checked through the use of a Cheeger–Simons class (cf. Dupont [8, Theorem 1.3]). This can then be combined with (A6) to give the following commutative diagram with exact rows and columns:



(A25)

We now turn our attention to (A19) and analyze the map from $H_*(G, C_1)$ to $H_*(G, Z_0)$ with the help of (A20). This analysis also works with $\tilde{G}$ in place of G. Let c be a k-cycle of $(T, \mathbb{Z})$. It is mapped onto the k-cycle $c \otimes (\infty, 0)$ of (G, C_1). The map $\partial : C_1 \to Z_0$ carries it to the k-cycle $c \otimes ((0) - (\infty))$ of (G, Z_0). The Weyl group generator $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ in G exchanges (0) and (∞) and induces the inversion automorphism of T. On the other hand, conjugation by w in G and simultaneous application of w to Z_0 induces the identity map on $H_*(G, Z_0)$. This shows that the image of the class of c in $H_k(G, Z_0)$ is 0 if $k > 0$ is even or if $k \equiv 1 \bmod 4$ and c is a torsion cycle. In the remaining cases, we compose with the homomorphism from $H_*(G, Z_0)$ to $H_*(G, C_0)$. With the identification in (A20), a similar argument shows that the composition amounts to multiplication by 2 if the class of c lies in $\Lambda^k_{\mathbb{Z}}(F^\times/\mu_F)$ with k odd or in $\mathbb{Q}/\mathbb{Z} \subset H_k(T, \mathbb{Z})$ with $k \equiv 3 \bmod 4$. This means that we have a kernel or order dividing 2 in the remaining cases. We therefore have the following commutative ladder with exact rows:



(A26)

As described in (A22), the map from $H_1(G, Z_0) \cong F^\times$ to $H_1(G, C_0) \cong F^\times$ corresponds to squaring. It follows that the map from $H_1(T, \mathbb{Z})$ to $H_1(G, Z_0)$ is an isomorphism and the map from $H_1(\tilde{T}, \mathbb{Z}) \cong F^\times$ to $H_1(\tilde{G}, Z_0) \cong F^\times$ must correspond to squaring and has kernel of order 2. We therefore have

$$\begin{aligned} &H_2(\tilde{G}, Z_0) \cong H_2(G, Z_0) \cong H_1(G, Z_1); \\ &H_1(\tilde{G}, Z_1) \cong H_1(G, Z_1) \amalg \mathbb{Z} \bmod 2, && \text{the projection on the first factor is induced by (A3);} \\ &\cong H_2(\tilde{G}, Z_0) \amalg \mathbb{Z} \bmod 2, && \text{the inclusion of the first factor is induced from a 'Bockstein'.} \end{aligned} \tag{A27}$$

To proceed further, we look at (A17). With (A13) at hand, (A17) can be viewed as a G-free resolution of the G-module Z_1. $H_*(G, Z_1)$ can therefore be determined directly by applying the functor $\mathbb{Z} \otimes_{\mathbb{Z}G} -$ to (A17) and taking the homology of the resulting chain complex. Using the fact that G is 3-transitive on $\mathbb{P}^1(F)$, it is immediate that $\mathbb{Z} \otimes_{\mathbb{Z}G} C_3 = \ker \partial$. As a consequence, we have the basic isomorphism

$$H_1(G, Z_1) \cong \mathbb{Z} \otimes_{\mathbb{Z}G} C_3 / \partial(\mathbb{Z} \otimes_{\mathbb{Z}G} C_4) = \mathscr{P}_F. \tag{A28}$$

We note that this isomorphism is valid over any field as long as G is taken to be $\mathrm{PGL}(2, F)$ and F has at least 4 elements.

We now combine (A27) and (A28) and substitute $\mathscr{P}_F$ for $H_2(\tilde{G}, Z_0)$ and $H_2(G, Z_0)$ in (A25). A careful tracing of the steps shows that this substitution uses only the assumption that $F^\times = F^{\times 2}$. The characteristic 0 assumption is used to check the injectivity of the map from $\mathbb{Q}/\mathbb{Z}$ to H_3 in (A25).

(A25) is essentially the desired theorem. As it stands $\tilde{\sigma}$ and σ determine each other. It is known that $\tilde{\sigma}$ carries $u \wedge v$ in $\Lambda^2_{\mathbb{Z}}(F^\times/\mu_F)$ onto $\{u, v\}^2$ in $K_2(F)$, see Milnor [14; Lemma 8.3, p. 65] or Sah–Wagoner [20; proof of Theorem 1.28, p. 629]. Here $u, v \in F^\times$, $\{u, v\}$ is the 'K_2-symbol'. We note that $u \in F^\times$ is mapped onto the matrix with u, u^{-1} on the diagonal in $\mathrm{SL}(2, F)$. It is therefore evident that the Bloch–Wigner theorem will follow if we show that $\varphi\{z\} = 2(z \wedge (1-z))$ holds for z in $F - \{0, 1\}$. This is the next task. (Note that the factor of 2 is immaterial because $\mathscr{P}_F$ is divisible (Theorem 5.1).)

We go back to (A12) and observe that $H_0(G, C_3)$ is just the free abelian group based on $\{z\}$ with $z \in F - \{0, 1\}$. Here $\{z\}$ corresponds to $(\infty, 0, 1, z)$ through the cross-ratio map. We have the following maps:

$$\begin{array}{ccccccc} H_0(G, C_3) & \xrightarrow{\partial} & H_0(G, Z_2) & \overset{l_1}{\cong} & H_1(G, Z_1) & \overset{l_2}{\cong} & H_2(G, Z_0) \\ & & & & & & \downarrow \\ H_2(T, \mathbb{Z}) & \xleftarrow{d_*} & H_2(B, \mathbb{Z}) & & \xleftarrow{\varrho} & & H_2(G, C_0) \end{array} \tag{A29}$$

l_1 results from the freeness of C_2 while l_2 results from (A27). ϱ is the inverse map resulting from Shapiro's lemma while d_* can be viewed as the map induced by the differential at 0 for the action of B on $\mathbb{P}^1(F) = F \cup (\infty)$, i.e. $d(h) = h(1) - h(0) \in F^\times$. Here we recall that $G = \mathrm{PGL}(2, F)$ so that $h \in F^\times$ is identified with $\left(\begin{smallmatrix} h & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Evidently, $l_1 \circ \partial$ is the natural projection onto $\mathscr{P}_F$ while α is the composition of the remaining maps starting from $H_2(G, Z_0)$. We will use α to denote the composition of all the maps in (A29).

For any left G-module M, let $C_*^{\mathrm{bar}}(G, M)$ denote the 'standard' normalized non-homogeneous complex so that $C_q(G, M)$ is generated by symbols $[g_1 | \cdots | g_q]x$, g_j in G, x in M, and boundary ∂_G is given by

$$\partial_G[g_1 | \cdots | g_q]x = [g_2 | \cdots | g_q]x + \sum_{1 \le i \le q-1} (-1)^i [g_1 | \cdots | g_i g_{i+1} | \cdots | g_q]x$$
$$+ (-1)^q [g_1 | \cdots | g_q] g_q(x).$$

We remark that our choice differs from the usual one by inversion in G, writing from right to left and a sign of $(-1)^q$. This has no effect on the homology groups, but will cause a difference in sign in the identification of the homology groups of an abelian group.

In order to define ϱ, we let $\hat{}: G \to G$ denote any section of $G \to G/B$ so that $\hat{g}_1 = \hat{g}_2$ holds if and only if $g_1^{-1} g_2$ lies in B. For $x_1, \ldots, x_q$, y in G, let $z_j = x_{j+1} \cdot \cdots \cdot x_q y$, $0 \le j \le q$. We then define

$$\varrho : C_*^{\mathrm{bar}}(G, \mathrm{ind}_B^G \mathbb{Z}) \longrightarrow C_*^{\mathrm{bar}}(B, \mathbb{Z}) \quad \text{with}$$
$$\varrho([x_1 | \cdots | x_q] yB) = [\hat{z}_0^{-1} x_1 \hat{z}_1 | \cdots | \hat{z}_{q-1}^{-1} x_q \hat{z}_q].$$

It is now straightforward to check that:

(A30) *ϱ is a chain map and induces an isomorphism from $H_*(G, \mathrm{ind}_B^G \mathbb{Z})$ to $H_*(B, \mathbb{Z})$ inverse to the inclusion of B into G.*

As before, $\mathrm{PGL}(2, F) = G$ acts on $\mathbb{P}^1(F)$ through fractional linear transformations. Fix z in $F - \{0, 1\}$ and let $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ be the Weyl group element in G. We make the following selections (depending on z):

$$g_1 = \begin{pmatrix} 0 & 1 \\ 1-z & z \end{pmatrix}, \qquad g_2 = \begin{pmatrix} z-1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad g_3 = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}. \tag{A31}$$

Define the section $\hat{}: G \to G$ to $G \to G/B$ according to the rule

$$\hat{g} = \begin{cases} 1 & \text{if } g(\infty) = \infty, \\ w & \text{if } g(\infty) = 0, \\ g_2 \circ w & \text{if } g(\infty) = 1, \\ g_2^2 \circ w & \text{if } g(\infty) = z. \end{cases} \tag{A32}$$

We compute

$$\begin{aligned}\partial\{z\} &= (0,1,z)-(\infty,1,z)+(\infty,0,z)-(\infty,0,1)=(g_1-g_2+g_3-1)(\infty,0,1)\\ &= \partial_G X_1 \quad \text{in } C_*^{\text{bar}}(G,C_2) \quad \text{and}\\ X_1 &= [g_2g_1^{-1}]g_1(\infty,0,1)-[g_3](\infty,0,1).\end{aligned} \tag{A33}$$

We obtain $l_1\circ\partial\{z\}$ by applying ∂ to X_1. Computing, we get

$$\begin{aligned}l_1\circ\partial\{z\} &= \partial X_1=\partial_G X_2 \quad \text{in } C_*^{\text{bar}}(G,C_1) \quad \text{and}\\ X_2 &= [g_3g_1g_2^{-1}\,|\,g_2g_1^{-1}](0,1)-[g_3g_2g_1^{-1}g_3^{-1}\,|\,g_3g_1g_2^{-1}](\infty,1)\\ &\quad -[g_3g_2g_1^{-1}g_3^{-1}g_1g_2^{-1}\,|\,g_2g_1^{-1}](0,z)+[g_3g_2g_1^{-1}g_3^{-1}g_1g_2^{-1}\,|\,g_2^2](\infty,0)\\ &\quad -[g_2^2\,|\,g_3^2](\infty,0)+[g_2g_1^{-1}\,|\,g_1^2](\infty,0)-[g_1^2\,|\,g_3^{-1}](\infty,0)+[g_3^{-1}\,|\,g_3^2](\infty,0).\end{aligned} \tag{A34}$$

To obtain (A34), we use

$$\begin{aligned}&g_1^{-1}g_3^{-1}g_1g_2=u_1=\text{multiplication by } z(z-1)=\left(\begin{smallmatrix}z(z-1) & 0\\ 0 & 1\end{smallmatrix}\right),\\ &g_2^{-2}g_3g_2=u_2=\text{multiplication by } z/(z-1)=\left(\begin{smallmatrix}z & 0\\ 0 & z-1\end{smallmatrix}\right),\\ &g_3g_2g_1^{-1}g_3^{-1}g_1g_2^{-1}=g_2^2g_3^2g_2^{-2},\\ &g_1^{-1}g_2^{-1}g_1^2=\text{multiplication by } z=g_3,\\ &g_2g_1^{-1}=g_1^2g_3^{-1}g_1^{-2}.\end{aligned} \tag{A35}$$

We obtain $l_2\circ l_1\circ\partial\{z\}$ by applying ∂ to X_2. This gives $\sum_{1\le i\le 16}(-1)^{i-1}[s_i\,|\,t_i](r_i)$, $s_i,t_i\in G$, r_i is $\infty,0,1$ or z. We can then apply ϱ in accordance with (A30), (A31) and (A32). After this, we apply d_*. Omitting the rather messy computations, we obtain:

In $H_2(T,\mathbb{Z})$, $\alpha\{z\}$ is represented by the following:

$$\begin{aligned}&[z-1\,|\,z^{-1}]-[z^{-2}\,|\,z]+[z^3\,|\,z^{-2}]-[z^{-1}\,|\,z-1]\\ &\quad+[z^2\,|\,z]-[z^{-2}\,|\,z]+[z^{-2}\,|\,1]-[z^2\,|\,(z-1)^2]\\ &\quad+[z\,|\,1]-[z^{-1}\,|\,z-1]+[z-1\,|\,z^{-1}]-[1\,|\,z]\\ &\quad+[(z-1)^2\,|\,z^2]-[1\,|\,z^{-2}]+[z\,|\,z^{-2}]-[z^{-1}\,|\,z^2].\end{aligned} \tag{A36}$$

As mentioned earlier, our choice of the bar complex requires us to identify the cell $[g_1\,|\cdots|\,g_q]g$ with $g^{-1}[g_q^{-1}\,|\cdots|\,g_1^{-1}]$. As a result, the identification of $H_2(T,\mathbb{Z})$ with $\Lambda^2_{\mathbb{Z}}(F^\times)$ is obtained by letting $a\wedge b$ denote the class of $[b^{-1}\,|\,a^{-1}]-[a^{-1}\,|\,b^{-1}]=$ class of $[b\,|\,a]-[a\,|\,b]$. Using the fact that $\partial_T[z^2\,|\,z\,|\,z^{-2}]=[z\,|\,z^{-2}]-[z^3\,|\,z^{-2}]+[z^2\,|\,z^{-1}]-[z^2\,|\,z]$, we obtain the desired result

$$\varphi\{z\}=2\cdot z\wedge(1-z)+z^{-1}\wedge z^2=2\cdot z\wedge(1-z). \tag{A37}$$

Remarks. 1. With (ii) of Corollary 5.2 at our disposal, we can look at the long homology exact sequence associated to the reduction mod p coefficient sequence.

Using the torsion-freeness of $H_2(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z}) \cong K_2(\mathbb{C})$ with the divisibility of $H_3(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z})$ we get $H_3(\mathrm{SL}(2,\mathbb{C}),\mathbb{F}_p) = 0$ for all p. This confirms Q5 on the level of H_3 for $\mathrm{SL}(2,\mathbb{C})$ as mentioned in the introduction. Notice that in accordance with Q5 the cohomology ring $H^*(\mathrm{SL}(2,\mathbb{C}),\mathbb{F}_p)$ is conjecturally a polynomial ring over $\mathbb{F}_p$ with a single generator c_2 in degree 4.

2. A careful analysis of the proof of the Bloch–Wigner theorem shows that only the divisibility of $K_2(\mathbb{C})$ is needed in obtaining (A5). This result already follows from the theorem of Matsumoto–Moore. Thus the divisibility of $\mathscr{P}_{\mathbb{C}}$ together with the Bloch–Wigner theorem (in the more precise version) give another proof the Bass–Tate theorem on the unique divisibility of $K_2(\mathbb{C})$. Of course, this argument works with $\mathbb{C}$ replaced by any algebraically closed field of characteristic 0. In the positive characteristic case, our line of argument would require a more careful treatment of (A11) – the center kills argument has to be replaced. However, we only need (A11) in low degrees and the center kills argument is still applicable when the characteristic is large enough. We omit the detailed analysis.

3. A variation of the theme of Bloch and Wigner can be carried out for $\mathrm{SO}(3,\mathbb{R})$. Call an $(i+1)$-tuple of points on $S(\mathbb{R}^3)$ 'independent' when any subset of size ≤ 3 is composed of linearly independent unit vectors in $\mathbb{R}^3$. The above complex (A12) can be replaced by the complex formed out of the independent simplices. A similar analysis can be carried out and we can obtain another proof of Mather's theorem. The relevant result needed is the identification of a suitable homology group with $\mathscr{P}(S(\mathbb{R}^2))/(\text{suspensions})$, hence with $\mathbb{R}/\mathbb{Z}$. In this sense, the geometric argument of Mather and the algebraic arguments of Matsumoto–Moore and Bass–Tate can be replaced by a common theme. However, the analogue of the divisibility of $\mathscr{P}_{\mathbb{C}}$ appears to be less clear. The difficulty lies with the lack of a good theory of 'invariants' for 4 independent points on $S(\mathbb{R}^3)$ under the action of $\mathrm{SO}(3,\mathbb{R})$. We will investigate this elsewhere.

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